Zero Energy Majorana Fermions in p-Wave Superfluids

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Introduction

p-wave superfluid/superconductor:
3He, p-wave Feshbach resonance, Sr_2RuO_4, Non-centrosymmetric SC

- **Ginzburg-Landau equation: Macroscopic scale physics** $\sim \xi$

  Spontaneous mass flow, textures, (Tsutsumi et al., Poster)

- **Quasiclassical Eilenberger equation: Intermediate region**

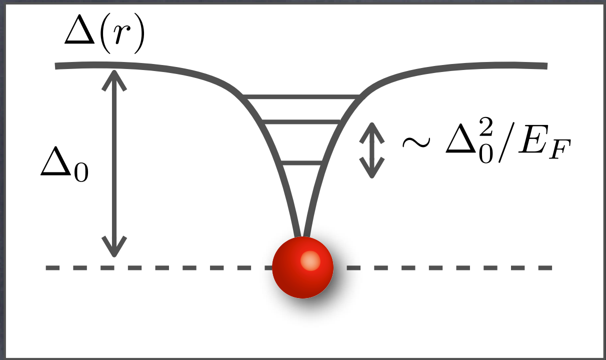
  Spontaneous mass flow and textures self-consistently determined from **quasiparticle states** (Ichioka et al.)
  Valid for the weak coupling regime $\xi \gg k_{\rm F}^{-1}$

- **Bogoliubov-de Gennes equation: Microscopic scale physics** $\sim \xi \quad \sim k_{\rm F}^{-1}$

  Quasiparticle excitations in quantum limit
  Applicable to the strong coupling regime beyond the BCS regime

  $\Rightarrow$ zero energy Majorana states appear in chiral p-wave SF's with vortices

Zero Energy States in Spin-Polarized p-Wave SF



Axisymmetric vortex in 2D s-wave pairing state

$$\Delta(\boldsymbol{r},\boldsymbol{k}) = \Delta(r)e^{iw\theta} \qquad w \in \mathbb{Z}$$

Core bound states in WEAK coupling limit

$$E_{\ell,n} = -\left(\ell - \frac{w}{2}\right)\epsilon_0 + \left(n - \frac{w-1}{2}\right)\epsilon_1$$

azimuthal quantum number $\ell \in \mathbb{Z}$ $\quad \epsilon_0 = \mathcal{O}\left(\frac{\Delta_0^2}{E_{\rm F}}\right), \quad \epsilon_1 = \mathcal{O}(\Delta_0)$

Axisymmetric vortex in "spinless" chiral p-wave state

$$\Delta(\boldsymbol{r},\boldsymbol{k}) = (k_x - ik_y)\Delta(r)e^{iw\theta}$$

$$E_{\ell,n} = -\left(\ell - \frac{w-1}{2}\right)\epsilon_0 + \left(n - \frac{w-1}{2}\right)\epsilon_1$$

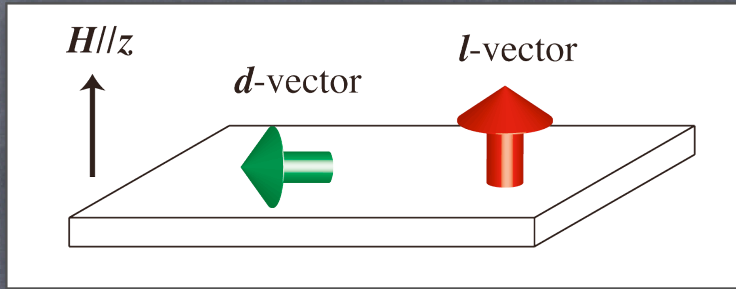
The lowest energy of the core-bound states

	w: odd	w: even
s-wave	Non-zero	Non-zero
p-wave	**zero**	Non-zero

Zero Energy States in "Spin-Triplet" p-Wave SF

e.g., 3He-A phase between parallel plates

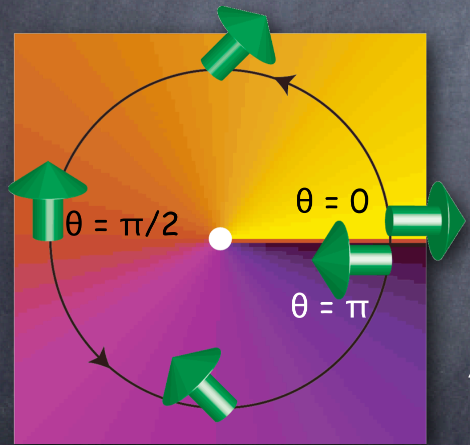
See, D. Ivanov, PRL **86**, 268 (2001)



- **Order parameter in 2D plate**

$$\Delta_\alpha(\boldsymbol{r},\boldsymbol{k}) = e^{iw\theta}\Delta(r)\hat{\boldsymbol{d}}_\alpha(\hat{k}_x + i\hat{k}_y)$$

Symmetry of d-vector $(\theta, \hat{\boldsymbol{d}}) \rightarrow (\theta+\pi, -\hat{\boldsymbol{d}})$



- **Fractional vortex** $w = \frac{1}{2}$

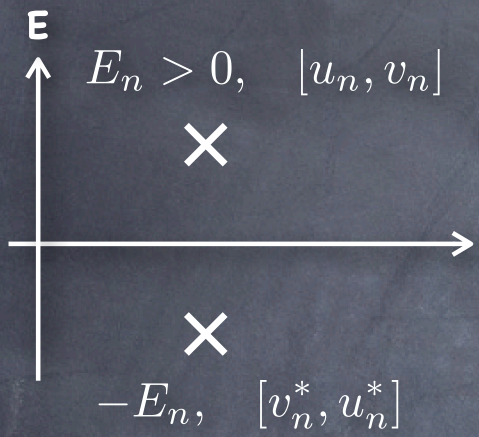
⇒ d-vector rotates in xy plane: **H**//z $\hat{d} = \frac{1}{\sqrt{2}}\left[\cos\frac{\theta}{2}\hat{x} + \sin\frac{\theta}{2}\hat{y}\right]$

$$\Delta(\boldsymbol{r},\boldsymbol{k}) = \frac{\Delta(r)}{2}\left[e^{i\theta}|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle\right](\hat{k}_x + i\hat{k}_y)$$

Singular vortex — Gapful

⇒ Low-energy excitation equivalent to the singular vortex of chiral k_x+ik_y pairing

Zero Energy Quasiparticles



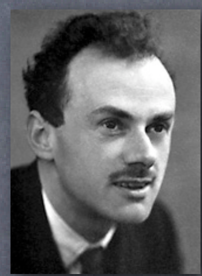
Bogoliubov quasiparticle $\Gamma_n^\dagger \equiv \int [u^*(\boldsymbol{r})\Psi^\dagger(\boldsymbol{r}) + v(\boldsymbol{r})\Psi(\boldsymbol{r})]\, d\boldsymbol{r}$

$$\Gamma_n^\dagger \neq \Gamma_n \qquad {}^\forall E_n \neq 0$$

Zero energy quasiparticle: consists of the equivalent contribution from the particle and hole

$$E_n = 0, \quad v^*_{E=0} = u_{E=0}$$

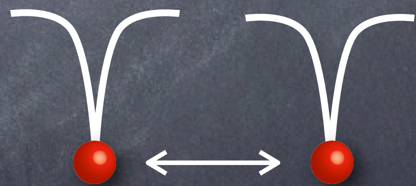
$$\Gamma^\dagger_{E=0} = \Gamma_{E=0}$$



self-conjugate operator = Majorana fermion

⇒ **Novel algebraic aspect (a type of non-abelian anyons)**

Aim



- Stability of zero energy Majorana fermions in p-wave superfluids with single vortex, especially, in atomic gases near p-wave Feshbach resonance
- Are they topologically protected against the vortex-vortex interaction?

Bogoliubov-de Gennes equation

$$\underline{\mathcal{M}}(x,y)\begin{bmatrix} u_n(x,y) \\ v_n(x,y)\end{bmatrix} = E_n \begin{bmatrix} u_n(x,y) \\ v_n(x,y)\end{bmatrix}$$

Diagonalization with {
- finite element method based on discrete variable representation
- shift-invert Lanczos/Arnoldi method

$$\mathcal{M}_{11}(x,y) = -\mathcal{M}_{22}(x,y) \equiv -\frac{\hbar^2}{2m}(\partial_x^2+\partial_y^2)-\mu$$

$$\mathcal{M}_{12}(\boldsymbol{r}) = -\mathcal{M}^*_{21}(\boldsymbol{r}) \equiv -\frac{1}{\sqrt{2}k_0}\left[\pm\Delta_{\pm 1}(\boldsymbol{r})\left(\partial_x \pm i\partial_y\right) \pm \frac{1}{2}\left(\partial_x \pm i\partial_y\right)\Delta_{\pm 1}(\boldsymbol{r})\right]$$

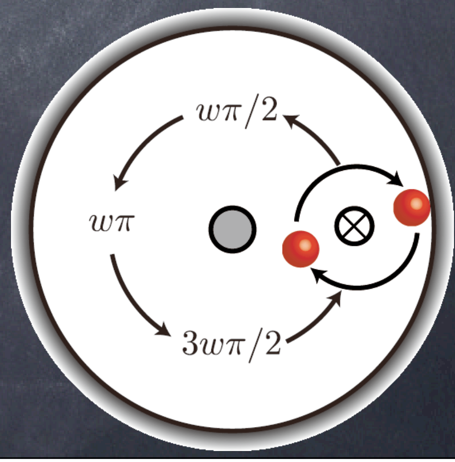
chiral p-wave state in 2D

Gap equation

Axisymmetric vortex state with odd winding number is assumed

$$\Delta_{\pm}(\boldsymbol{r}) = \frac{g}{\sqrt{2}k_0}\sum_{E_n<0}\left[v_n^*(\boldsymbol{r})(\partial_x \pm i\partial_y)u_n(\boldsymbol{r}) - u_n(\boldsymbol{r})(\partial_x \pm i\partial_y)v^*_n(\boldsymbol{r})\right]$$

The logarithmic divergence on the cutoff energy can not be removed by replacing "g" to renormalized one



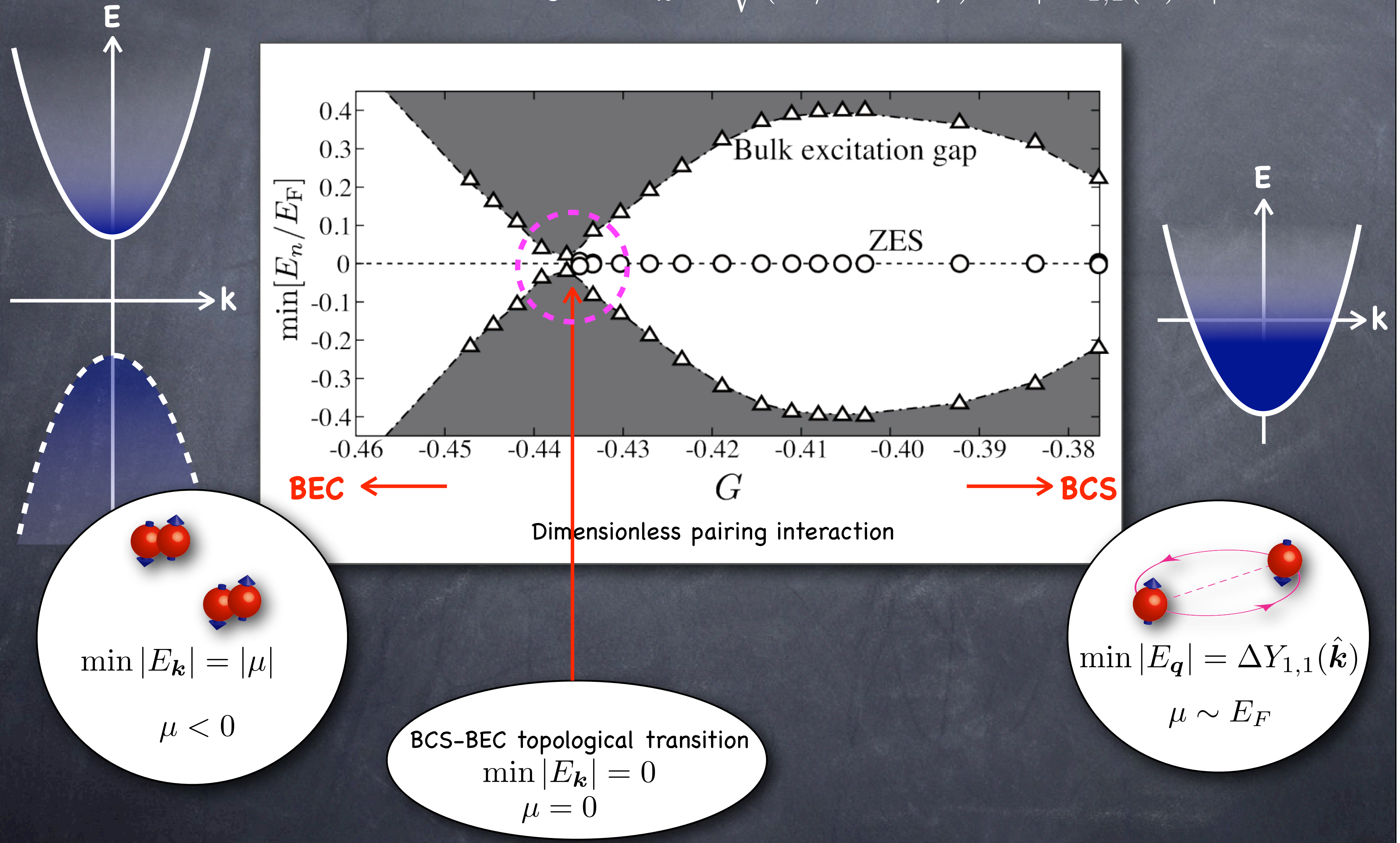
Conservation of particle number

$$N = \sum_n \int |u_n(\boldsymbol{r})|^2 d\boldsymbol{r}$$

BEC $\mu < 0$ $\longleftrightarrow$ $\mu \sim E_{\rm F}$ BCS

Low-Energy Spectrum near p-Wave Resonance

Bulk excitation energy $E_{\boldsymbol{k}} = \sqrt{(k^2/2M - \mu)^2 + |kY_{1,1}(\hat{\boldsymbol{k}})\Delta|^2}$



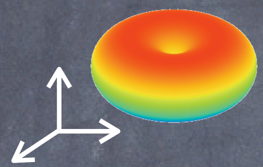
Dispersion of Edge States

e.g., Stone and Roy, PRB '04

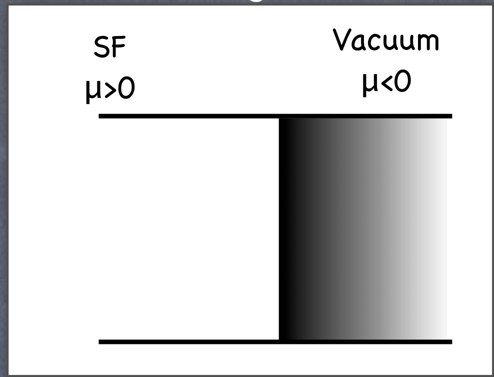
Edge states

Andreev bound states at the edge

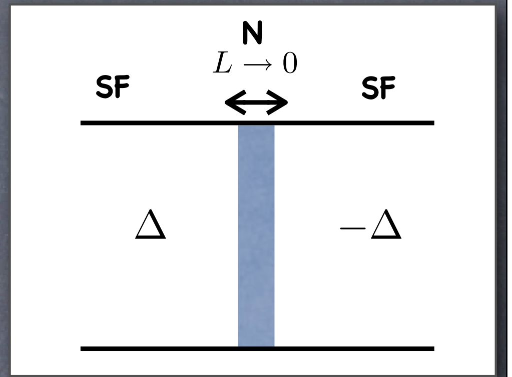
$$\Delta(\boldsymbol{r},\boldsymbol{k}) \propto k_x - ik_y$$



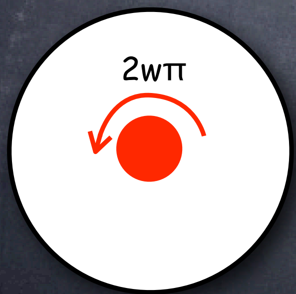
QP scattering at the wall



1D S/N/S junction

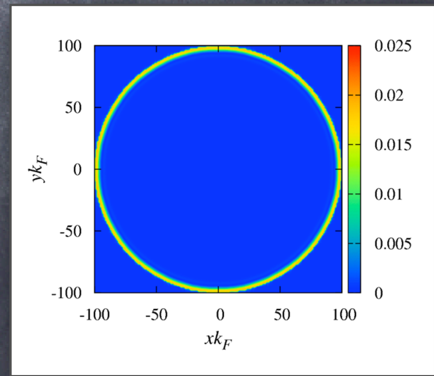


Axisymmetric chiral p-wave state with winding number w



$$\Delta(\boldsymbol{r},\boldsymbol{k}) = \frac{1}{\sqrt{2}k_0}(k_x - ik_y)\Delta(r)e^{iw\theta}$$
$$u_Q(\boldsymbol{r}) = u_Q(r)e^{iQ\theta}$$
$$v_Q(\boldsymbol{r}) = v_Q(r)e^{i(Q-w-1)\theta}$$

$$E_Q = \left(Q - \frac{w-1}{2}\right)\frac{\Delta}{k_F R} \qquad Q \in \mathbb{Z}$$



Zero energy "edge" state appears if w is ODD

Weak coupling BCS regime

$$\Delta_0/E_{\rm F} = 0.3, \quad \mu \sim E_{\rm F}$$

The wave function is bound at the core and edge

2 length scale $\quad \xi > k_{\rm F}^{-1}$

$$E/E_{\rm F} = \mathcal{O}(10^{-15})$$

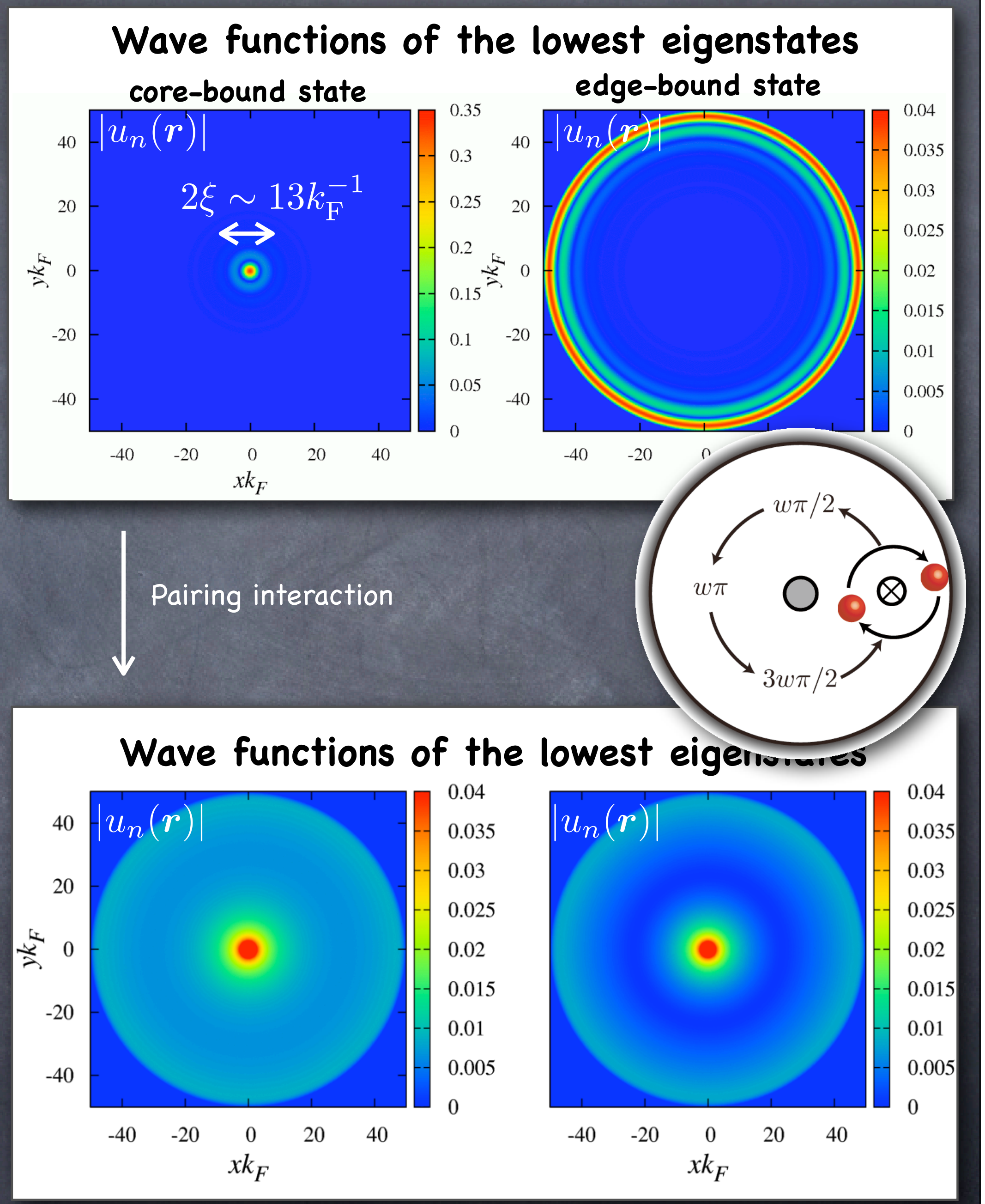
doubly degenerate
zero energy states

BCS-BEC transition point

$$\Delta_0/E_{\rm F} = 0.65, \quad \mu \sim 0$$

Spatial variation having long wavelength leads to the overlap between edge- and core-wave functions.

$$E_n/E_{\rm F} = \pm 0.007$$



Summary on Single Vortex

- Quasiparticle excitations with zero energy appear if chiral p-wave superfluids have single vortex with ODD winding number
- The zero energy "Majorana" fermions are bound at the vortex core and edge in the weak coupling BCS regime.
- The Majorana fermions survive until the BCS-BEC transition point, and vanish in the BEC phase (gapful excitation), where the low energy excitation in BEC regime is trivial, which is determined by the chemical potential.
- At the transition point, the spatial overlap between core- and edge-bound wavefunctions gives rise to the splitting of the degenerate zero energy eigenstates.

-TM, Ichioka, Machida, PRL **101**, 150409 (2008)

-Tsutsumi, Kawakami, TM, Ichioka, Machida, PRL **101**, 135302 (2008)

Non-Abelian Braiding Statistics

Ivanov, PRL **86**, 268 (2001)

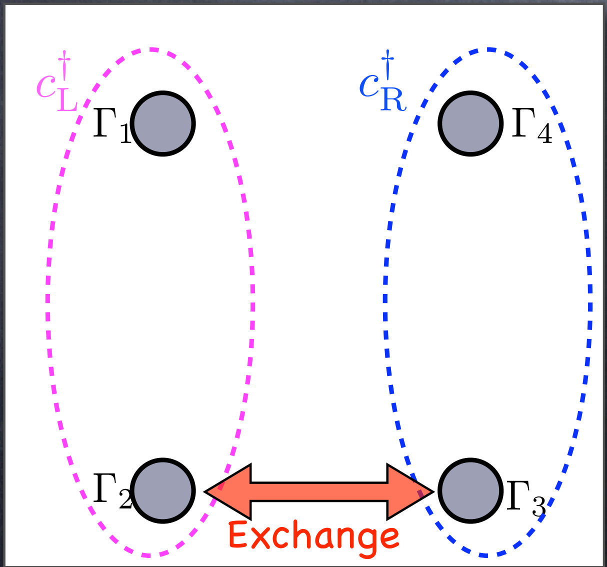
Zero energy Majorana fermions $\Gamma^\dagger = \Gamma$

Complex fermion $c \equiv \frac{1}{2}(\Gamma_1 + i\Gamma_2)$ $c^\dagger \neq c$

Vacuum of complex fermions $|0\rangle$

occupied state $|1\rangle \equiv c^\dagger|0\rangle$

$\Gamma_1, \quad \Gamma_2, \quad \Gamma_1\Gamma_2 \quad \Rightarrow \quad \sigma_x, \quad \sigma_y, \quad \sigma_z$ Pauli matrices



Non-Abelian statistics of vortices

Degenerate ground states:

$$|0\rangle_L|0\rangle_R, \ |1\rangle_L|1\rangle_R, \ |1\rangle_L|0\rangle_R, \ |0\rangle_L|1\rangle_R$$

"Exchange"

$$|1\rangle_L|1\rangle_R \ \mapsto \ |1\rangle_L|1\rangle_R + b|0\rangle_L|0\rangle_R$$

⇒ Pair annihilation of zero energy Majorana fermions

Braiding vortices ⇒ "Quantum circuit"

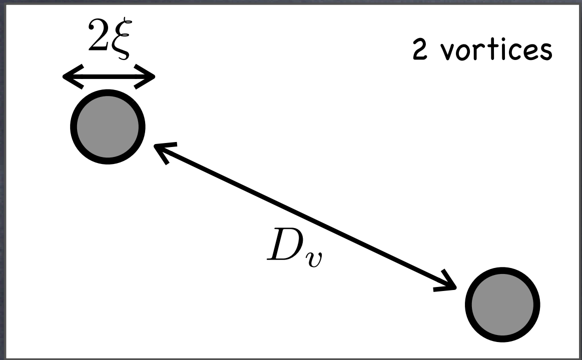
Kitaev, Ann. Phys. **303**, 2 (2003)

Freedman et al., Commun. Math. Phys. **227**, 605 (2003)

Questions

The zero energy states are topologically protected (?)

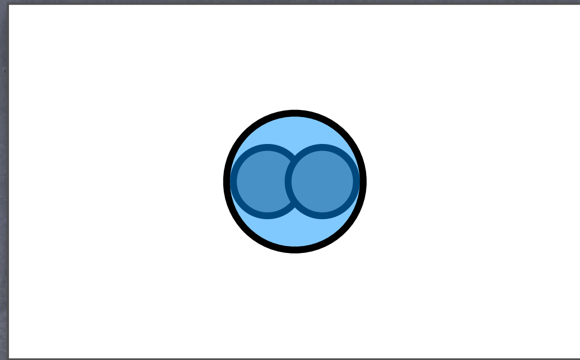
Dilute limit $D_v \gg \xi$



?Intermediate region?

$D_v \sim \mathcal{O}(10\xi)$

Dense limit $D_v \leq \xi$



Each vortex involve zero energy states

Giant vortex with EVEN winding number

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No zero energy excitations!

Aim

Robustness of zero energy states against the **vortex interaction** & external disturbances

e.g., **vortex distance D_v**, vortex number N_v, pinning potential, ...

Setting up the Model

$$\Delta(\boldsymbol{r},\boldsymbol{k}) = \frac{1}{\sqrt{2}k_0}(k_x - ik_y)\prod_{j=1}^{N_v} e^{i\bar{\theta}_j} f(\bar{r}_j)$$

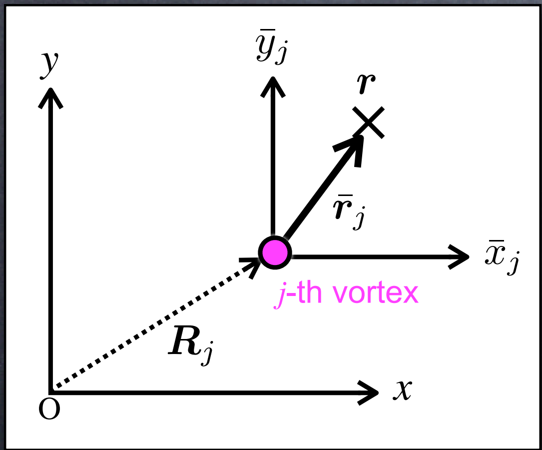
Orbital motion (chiral state)

Center-of-mass motion (Vortices)

$$f(\bar{r}_j) \equiv \Delta_0 \tanh\left(\frac{|\boldsymbol{r}-\boldsymbol{R}_j|}{\xi}\right)$$

Number of vortices: N_v

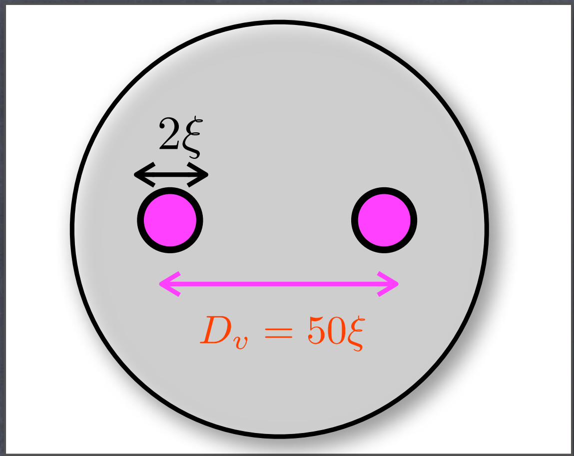
Distance between vortices: D_v

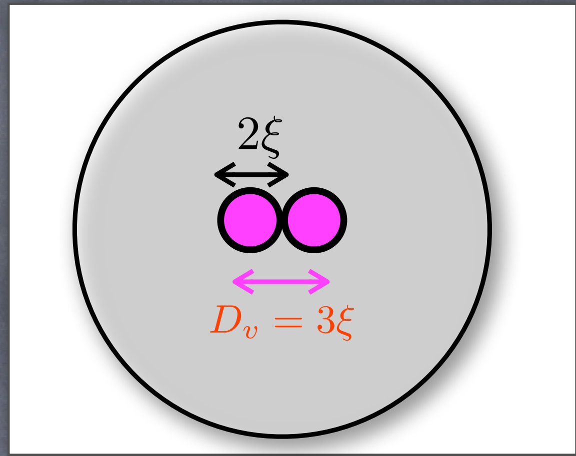


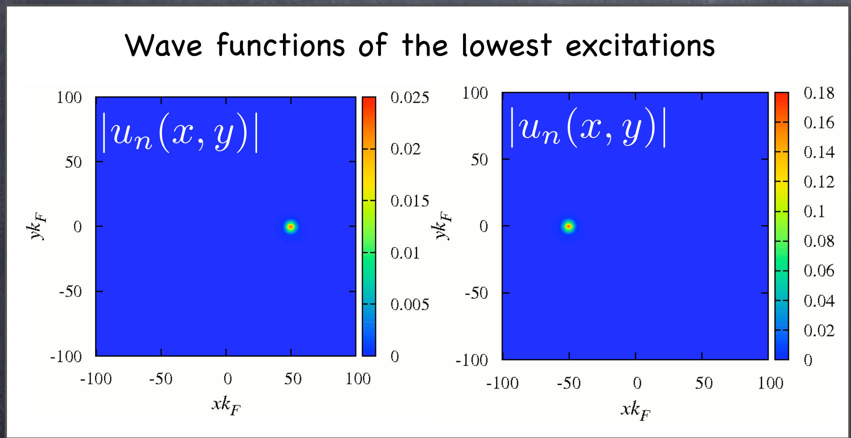
Bogoliubov-de Gennes equation

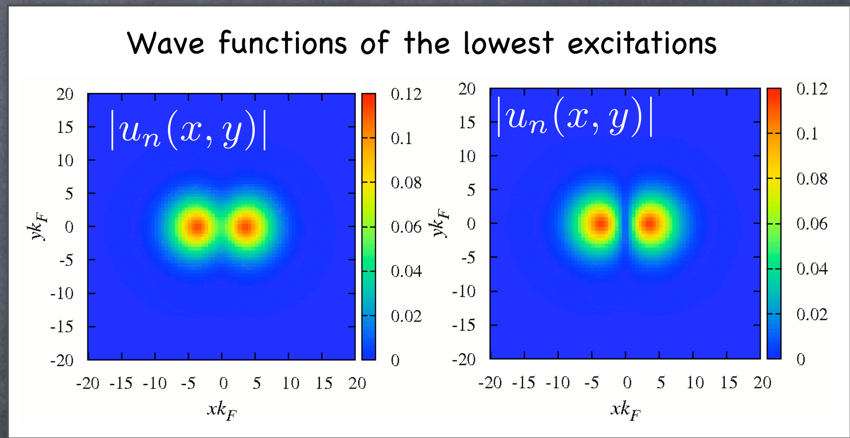
$$\underline{\mathcal{M}}(x,y)\begin{bmatrix} u_n(x,y) \\ v_n(x,y) \end{bmatrix} = E_n \begin{bmatrix} u_n(x,y) \\ v_n(x,y) \end{bmatrix}$$

2-Vortex System









Separated core-bound state with zero energy

$$E/E_F = \mathcal{O}(10^{-16})$$

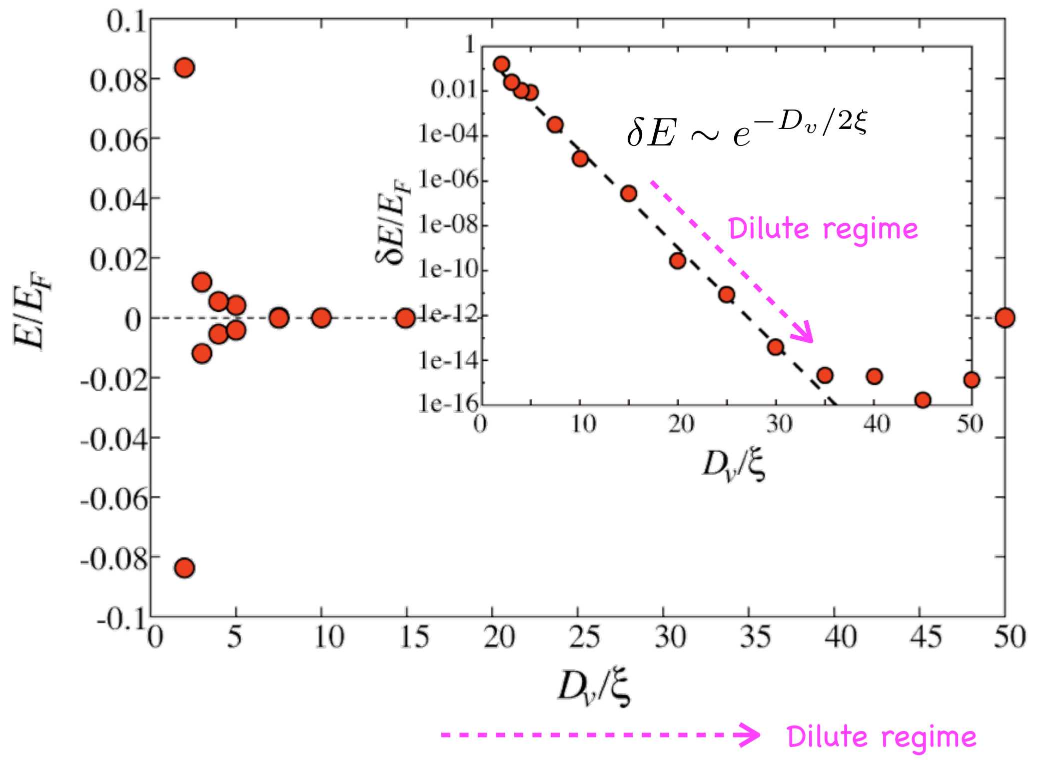
Majorana fermions

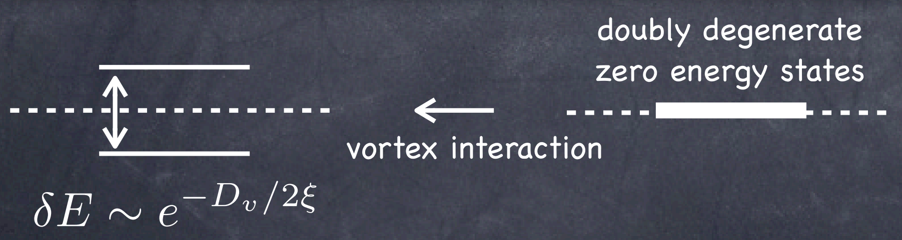
$E = 0.0119E_F$ $\quad$ $E = -0.0119E_F$

Overlap between core-bound states
⇒ bonding and anti-bonding states

Doubly degenerate zero energy states appears in the DILUTE LIMIT,
which are exponentially shifted from zero as the vortex distance (Dv) becomes narrow

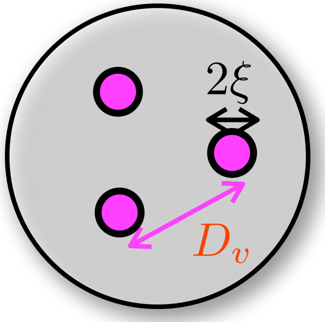
Splitting of eigenenergies of the lowest core-bound states



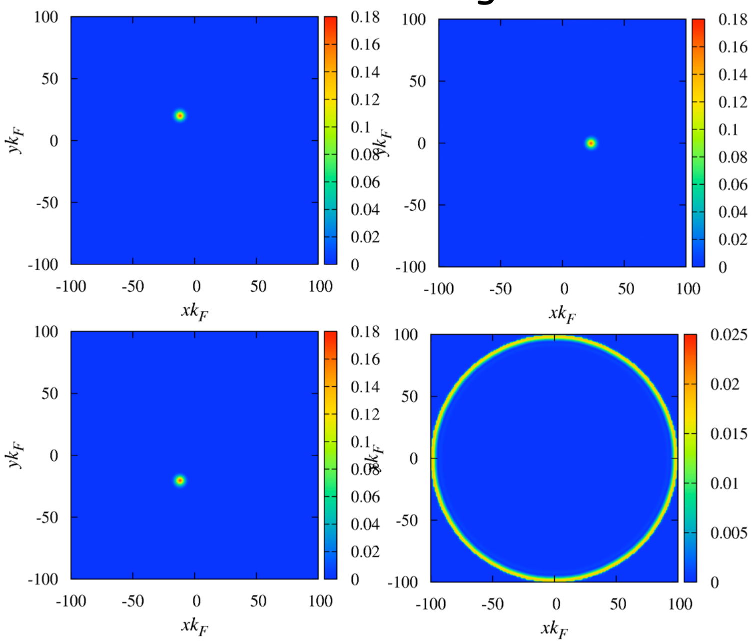


3-Vortex System

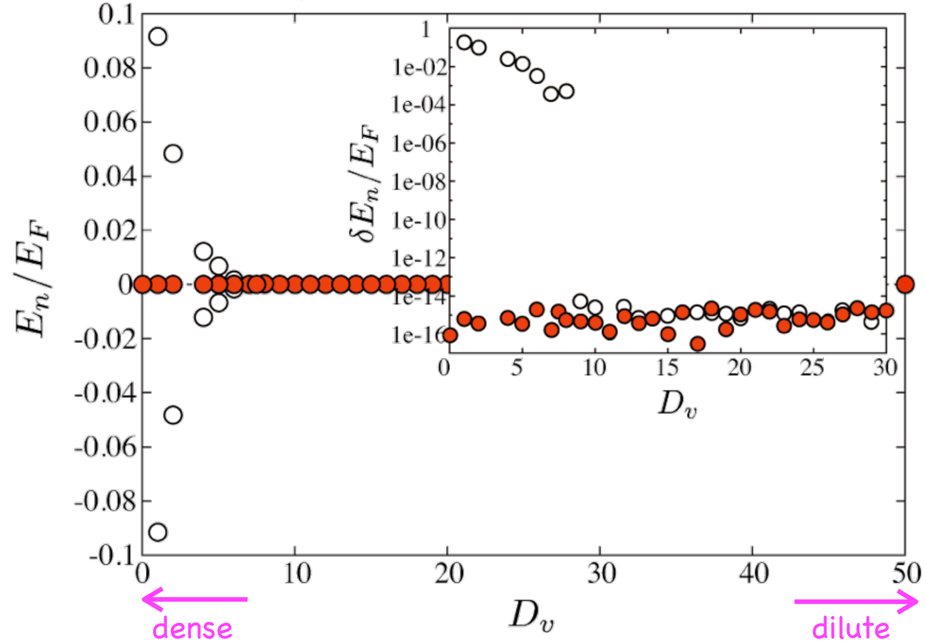
Vortex configuration

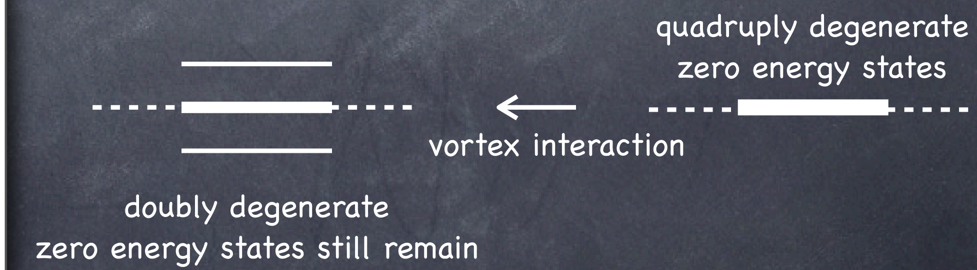


Wave functions of zero energy states in the dilute regime

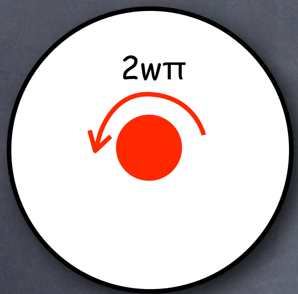


Splitting of eigenenergies of core-bound states





Protection of Zero Energy States

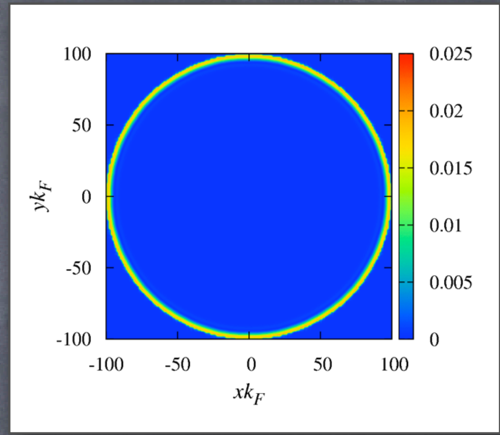


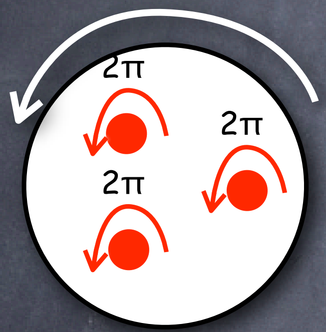
In an axisymmetric vortex with Nv=1

$$u_Q(\boldsymbol{r}) = u_Q(r)e^{iQ\theta}$$
$$v_Q(\boldsymbol{r}) = v_Q(r)e^{i(Q-w-1)\theta}$$

$$E_Q = \left(Q - \frac{w-1}{2}\right)\frac{\Delta}{k_F R} \qquad Q \in \mathbb{Z}$$

$\Rightarrow$ Zero energy "edge" state appears **if w is odd**

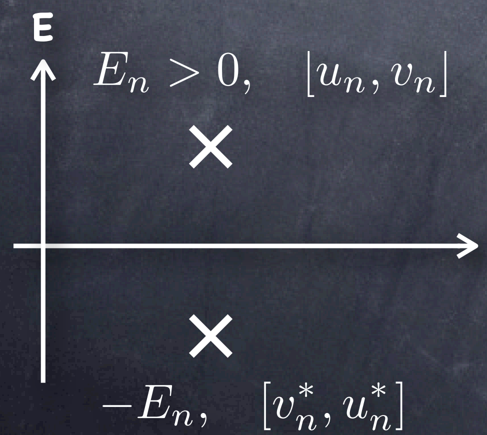




✓In the case of Nv > 1

The lowest edge state in odd Nv systems always have zero energy where the robustness is independent of Dv (odd winding number is not necessary).

$$E_Q = \left(Q - \frac{N_v-1}{2}\right)\frac{\Delta}{k_F R} \qquad Q \in \mathbb{Z}$$



✓Particle-hole symmetry of BdG eigenstates

The total number of BdG eigenstates is mod 2

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in addition to the edge state, the another zero energy state must appear if Nv is ODD.

Concluding Remarks

Robustness of zero energy "Majorana" states in chiral p-wave SF with many vortices

1. Even number Nv

✓The zero energy states appear in the dilute limit, whose wave functions are bounded at the core

✓The eigenenergies of the lowest core-bound states are exponentially lifted from the zero as vortex interaction increases

2. Odd number Nv

✓Zero energy states bound at CORE and EDGE appear in the dilute limit

✓Two of them always remain zero energy, and the robustness is independent of vortex distance and external disturbances

The full self-consistent calculation qualitatively reproduces the results obtained by the test potential ansatz

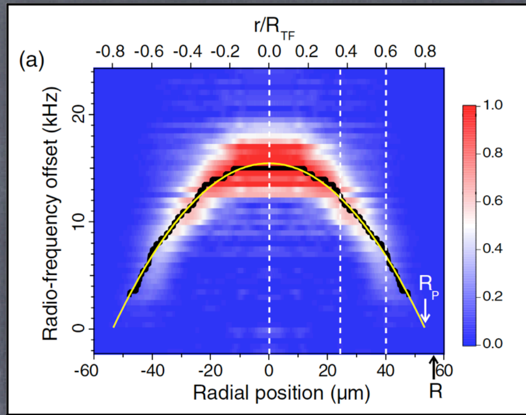
-TM and Machida, in preparation

Remaining problems

✓How to observe the "Majorana" fermions in atomic gases and 3He-A

--The zero energy states may be detectable with the spatially resolved rf measurement, i.e., the probe of LDOS.

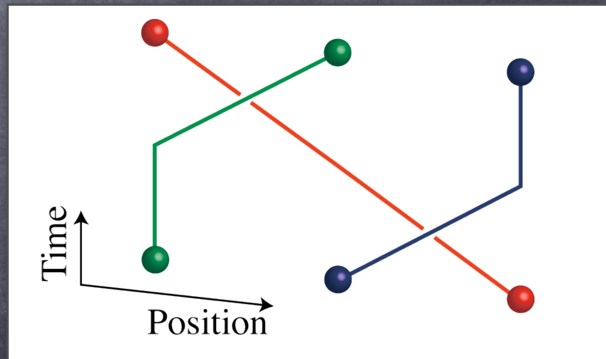
Y. Shin et al., PRL **99**, 090403 (2008)



✓Microscopic simulation of braiding vortices with the zero energy quasiparticles

Degenerate ground states $\Psi_1, \cdots, \Psi_N$

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \vdots \\ \Psi_N \end{bmatrix} \longmapsto \mathcal{U} \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \vdots \\ \Psi_N \end{bmatrix}$$



universal matrix determined by vortex exchange