Non-Abelian Vortices in Spinor Bose-Einstein Condensates

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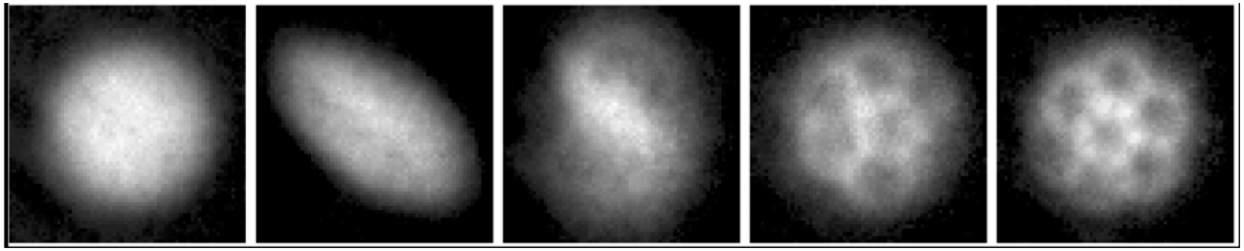
Vortices in Bose-Einstein Condensates



vortex in ^{87}Rb BEC

K. W. Madison et al.
PRL 86, 4443 (2001)

vortex in ^{4}He

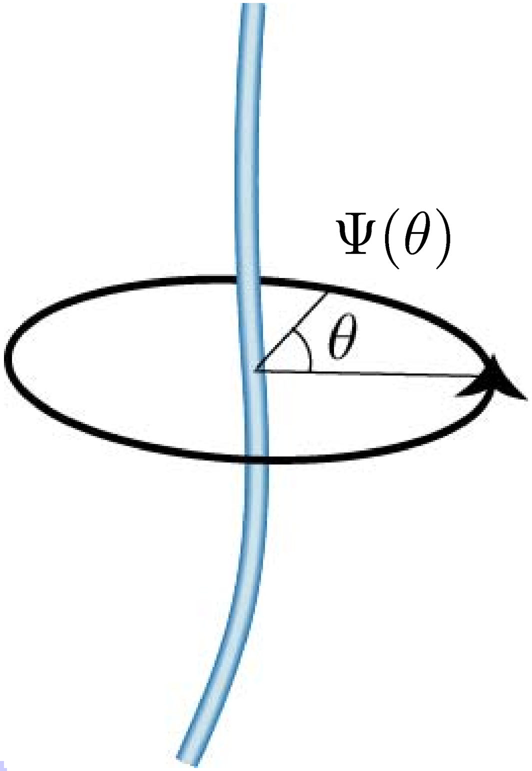
G. P. Bewley et al.
Nature 441, 588 (2006)

Vortices appears as line defects when symmetry breaking happens

- Vortices are Abelian for single-component BEC

Quantized Vortex and Topological Charge

Topological charge of a vortex can be considered how order parameter changes around the vortex core



Single component BEC : $\Psi(\theta) \propto \exp[in\theta]$

Topological charge can be expressed by integer n

Quantized Vortex and Topological Charge

Topological charge of a vortex can be considered how order parameter changes around the vortex core

Topological charge can be expressed by the first homotopy group

single component BEC
¼$_1$(G/H) = Z

G (= U(1)) : Symmetry of the system
H (= 1) : Symmetry of the order-parameter

When topological charge can be expressed by non-commutative algebra (: first homotopy group ¼$_1$ is non-Abelian), we define such vortices as "**non-Abelian vortices**"

Quantized Vortex and Topological Charge

Topological charge of a vortex can be considered how order parameter changes around the vortex core

Non-Abelian vortices can be realized in the cyclic phase of spin-2 Bose Einstein condensates.

Introduction of spinor BEC

Hamiltonian of spinor boson system (without trapping and magnetic field)

$$H = -\int d\boldsymbol{x}\, \frac{\hbar^2}{2M} \nabla\Psi^\dagger_m(\boldsymbol{x})\nabla\Psi_m(\boldsymbol{x})$$

$$+\frac{1}{2}\int d\boldsymbol{x}_1 \int d\boldsymbol{x}_2 \Psi^\dagger_{m_1}(\boldsymbol{x}_1)\Psi^\dagger_{m_2}(\boldsymbol{x}_2)V_{m_1 m_2 m'_1 m'_x}(\boldsymbol{x}_1-\boldsymbol{x}_2)\Psi_{m'_2}(\boldsymbol{x}_2)\Psi_{m'_1}(\boldsymbol{x}_1)$$

Contact interaction (l $= 0$)

$$V_{m_1 m_2 m'_1 m'_x}(\boldsymbol{x}_1-\boldsymbol{x}_2) = \delta(\boldsymbol{x}_1-\boldsymbol{x}_2)\sum_{F=even} g_F P_F$$

$$P_F = \sum_{m_1,m_2,m'_1,m'_2,M} O^{F,M}_{m_1 m_2}\left(O^{F,M}_{m'_1 m'_2}\right)^* |F,m'_1\rangle \otimes |F,m'_2\rangle\langle F,m_2| \otimes \langle F,m_1|$$

Mean Field Approximation for BEC at $T = 0$

Case of Spin-2

$$H \simeq \int d\boldsymbol{x} \left[-\Psi_m^* \frac{\hbar^2}{2M} \nabla^2 \Psi_m + \frac{c_0}{2} n_{\mathrm{tot}}^2 + \frac{c_1}{2} \boldsymbol{F}^2 + \frac{c_2}{2} |A_{00}|^2 \right]$$

$$c_0 = \frac{4g_2 + 3g_4}{7}, \quad c_1 = \frac{g_4 - g_2}{7}, \quad c_2 = \frac{7g_0 - 10g_2 + 3g_4}{7}$$

$$n_{\mathrm{tot}}(\boldsymbol{x}) = \Psi_m^*(\boldsymbol{x}) \Psi_m(\boldsymbol{x}), \quad \boldsymbol{F}(\boldsymbol{x}) = \Psi_m^*(\boldsymbol{x}) \hat{\boldsymbol{F}}_{mm'}(\boldsymbol{x}) \Psi_{m'}(\boldsymbol{x})$$

$$A_{00}(\boldsymbol{x}) = \frac{1}{\sqrt{5}} [2\Psi_2(\boldsymbol{x})\Psi_{-2}(\boldsymbol{x}) - 2\Psi_1(\boldsymbol{x})\Psi_{-1}(\boldsymbol{x}) + \Psi_0(\boldsymbol{x})^2]$$

n_{tot} : total density
F : magnetization
A_{00} : singlet pair amplitude

Spin-2 BEC

$$H \simeq \int d\boldsymbol{x} \left[-\Psi_m^* \frac{\hbar^2}{2M} \nabla^2 \Psi_m + \frac{c_0}{2} n_{\rm tot}^2 + \frac{c_1}{2} \boldsymbol{F}^2 + \frac{c_2}{2} |A_{00}|^2 \right]$$

1. $c_1 < 0 \longrightarrow$ ferromagnetic phase : $F \neq 0$
2. $c_1 > 0,\ c_2 < 0 \longrightarrow$ polar phase : $F = 0,\ A_{00} \neq 0$
3. $c_1 > 0,\ c_2 > 0 \longrightarrow$ cyclic phase : $F = A_{00} = 0$

ferromagnetic

$$e^{i\phi} e^{-i\boldsymbol{e}\cdot\hat{\boldsymbol{F}}\alpha} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

polar

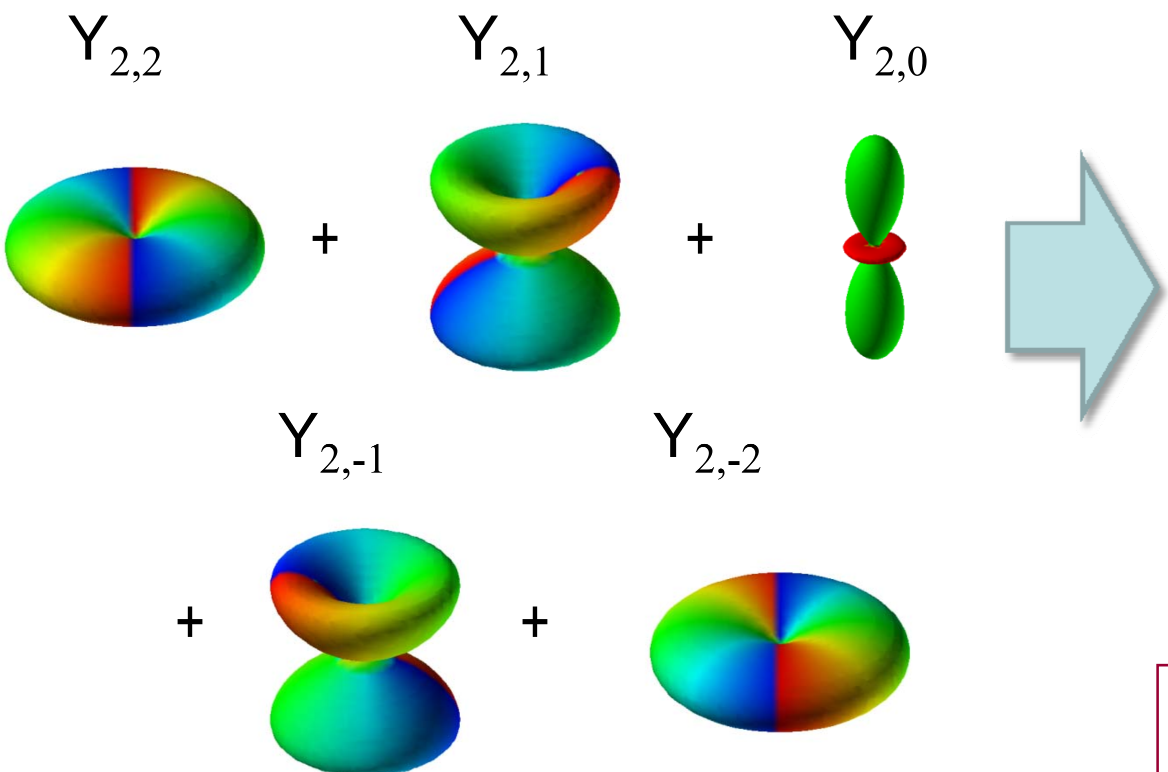
$$e^{i\phi} e^{-i\boldsymbol{e}\cdot\hat{\boldsymbol{F}}\alpha} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

cyclic

$$e^{i\phi} e^{-i\boldsymbol{e}\cdot\hat{\boldsymbol{F}}\alpha} \begin{pmatrix} i/2 \\ 0 \\ 1/\sqrt{2} \\ 0 \\ i/2 \end{pmatrix}$$

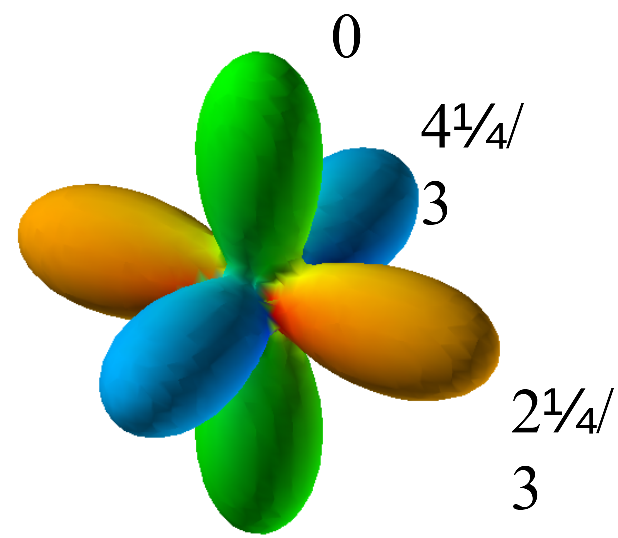
Spin-2 BEC

$$\sum_{m=-2}^{2} \Psi_m Y_{2,m}$$



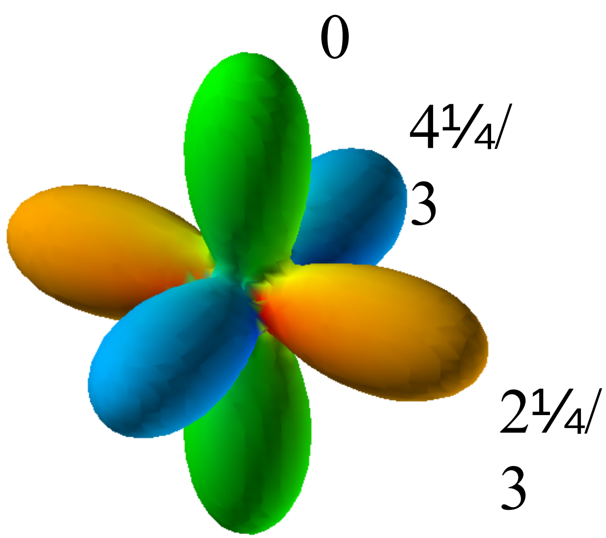
Cyclic phase

$$e^{i\phi}(3\cos^2\theta + \sqrt{3}i\sin^2\theta\cos 2\varphi - 1)$$

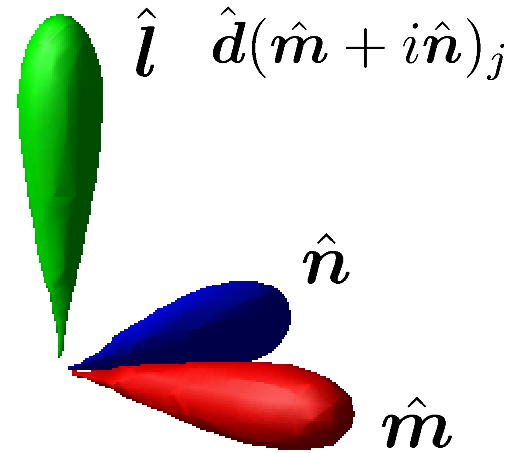


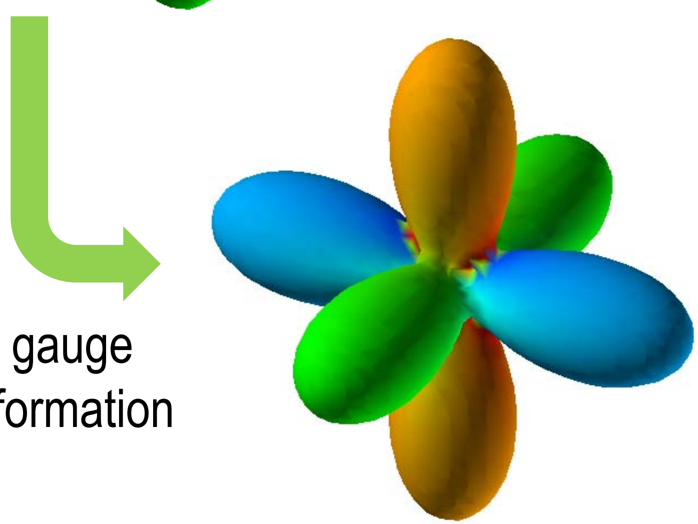
headless triad

Triad of ^{3}He-A and cyclic phase



^{3}He-A





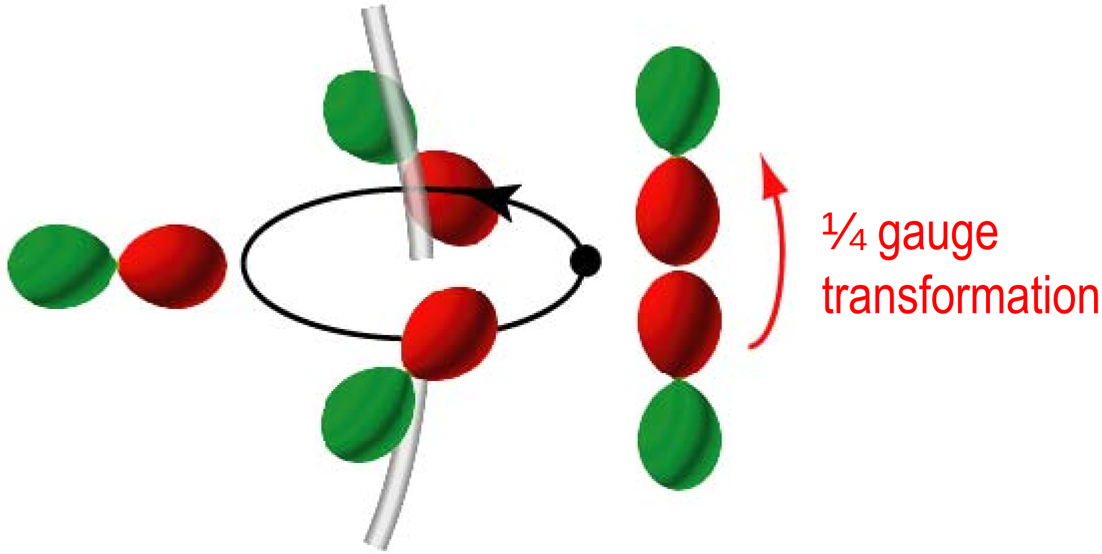
2¼/3 gauge transformation

1. Having a ¼–rotational symmetry
2. Three axes can interchange each other by 2¼/3 gauge transformation

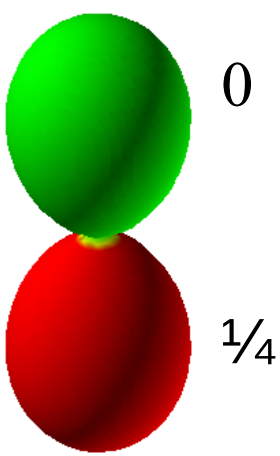
Vortices in Spinor BEC

S = 1 Polar phase

$$e^{i\phi}e^{-i\boldsymbol{e}\cdot\hat{\boldsymbol{F}}\alpha}\begin{pmatrix}0\\1\\0\end{pmatrix}$$



headless vector



Half quantized vortex : spin & gauge rotate by ¼ around vortex core

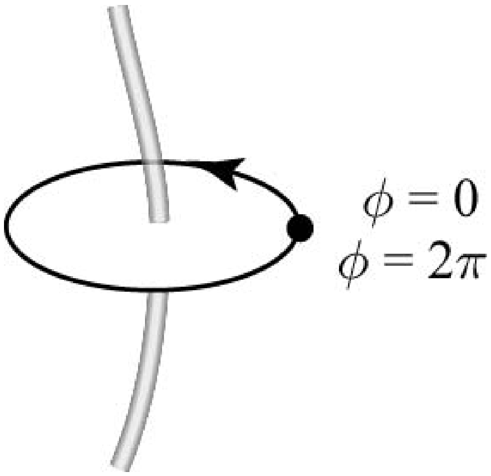
Topological charge can be expressed by integer and half integer (Abelian vortex)

$$\pi_1(G/H) = Z_2 \ltimes Z$$

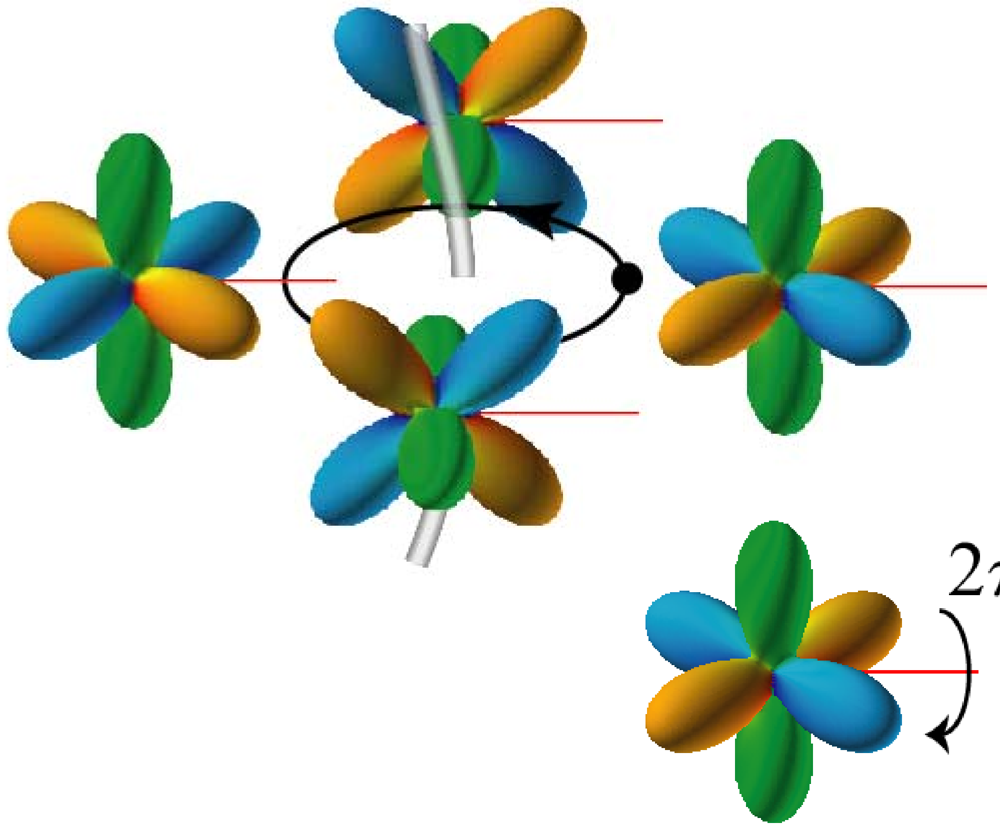
Vortices in Spin-2 BEC

There are 5 types of vortices in the cyclic phase

gauge vortex

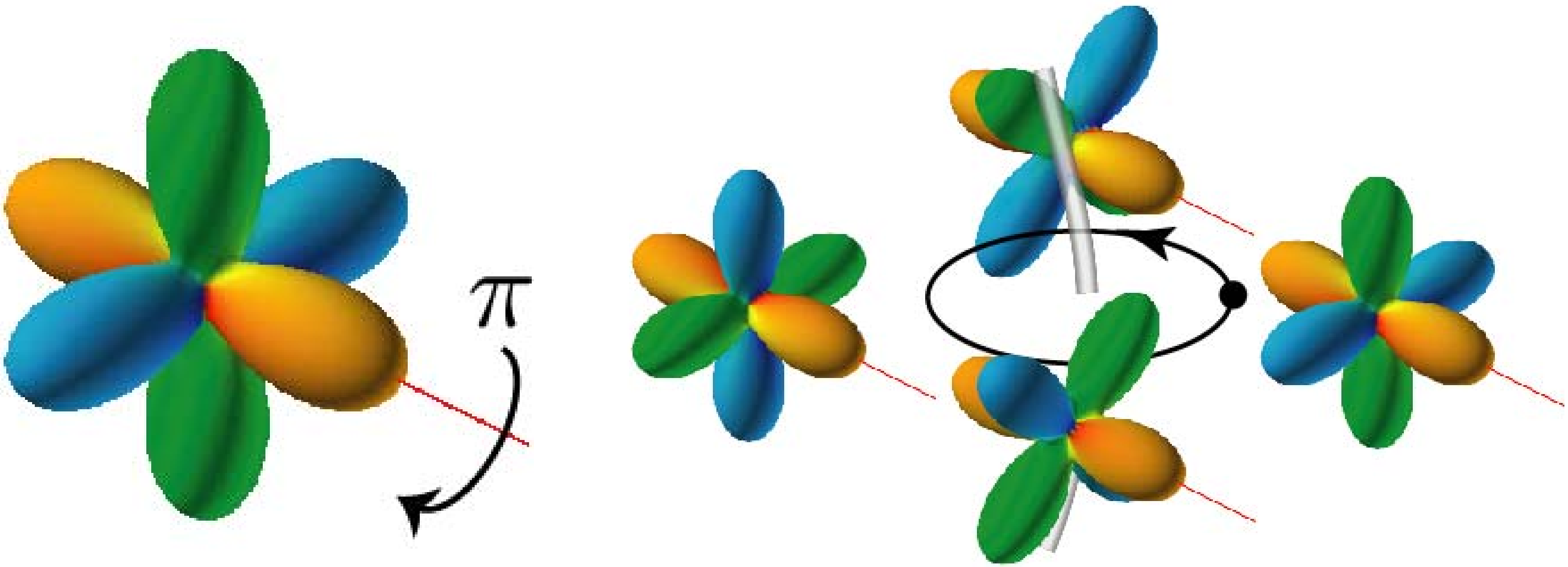


integer spin vortex



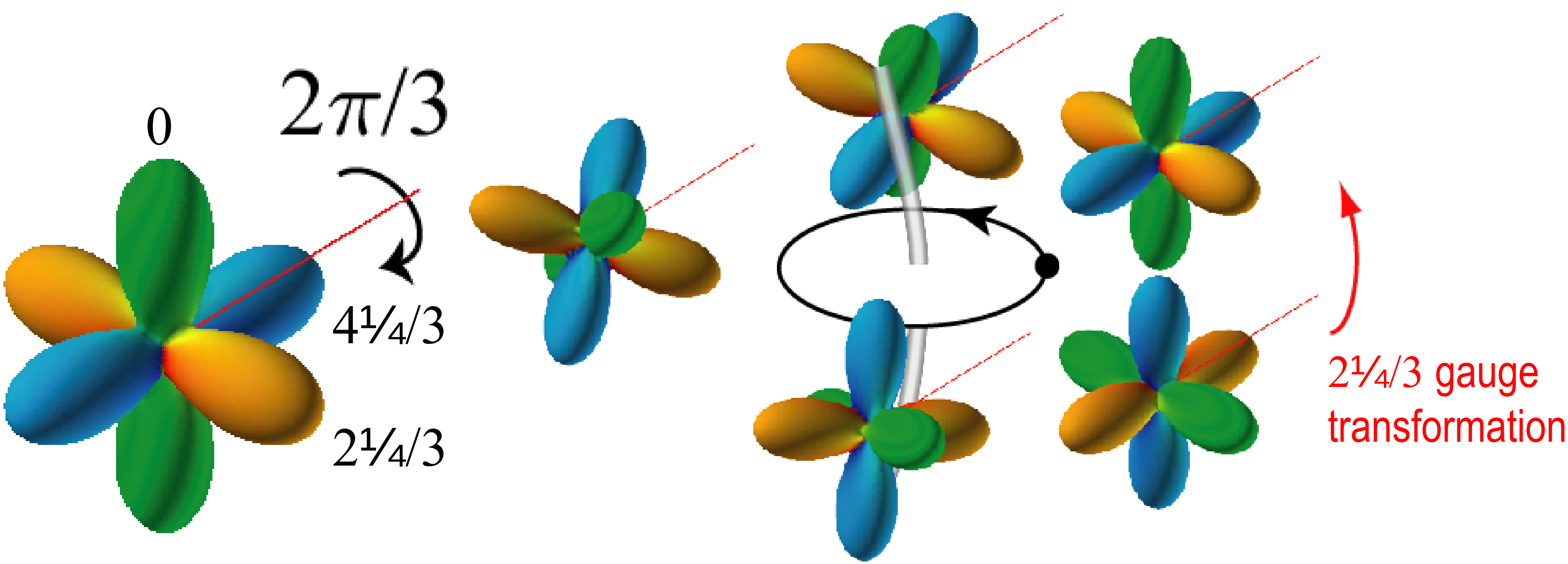
Vortices in Spin-2 BEC

1/2-spin vortex : triad rotate by ¼ around three axis e_x, e_y, e_z



Vortices in Spin-2 BEC

1/3 vortex : triad rotate by 2¼/3 around four axis e_1, e_2, e_3 , e_4
and 2¼/3 gauge transformation

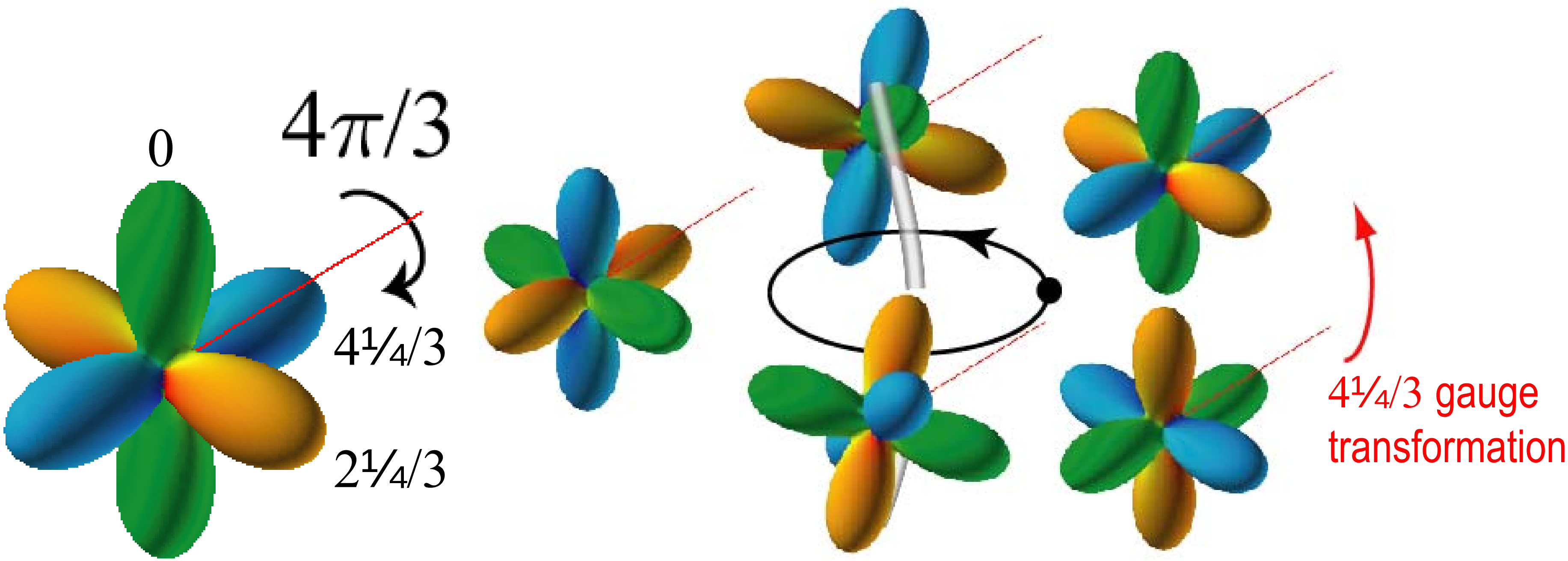


$$\boldsymbol{e}_1 = (1,1,1)/\sqrt{3}, \quad \boldsymbol{e}_2 = (1,-1,-1)/\sqrt{3}$$

$$\boldsymbol{e}_3 = (-1,1,-1)/\sqrt{3}, \quad \boldsymbol{e}_4 = (-1,-1,1)/\sqrt{3}$$

Vortices in Spin-2 BEC

4, 2/3 vortex : triad rotate by 4¼/3 around four axis e_1, e_2, e_3 , e_4 and 4¼/3 gauge transformation



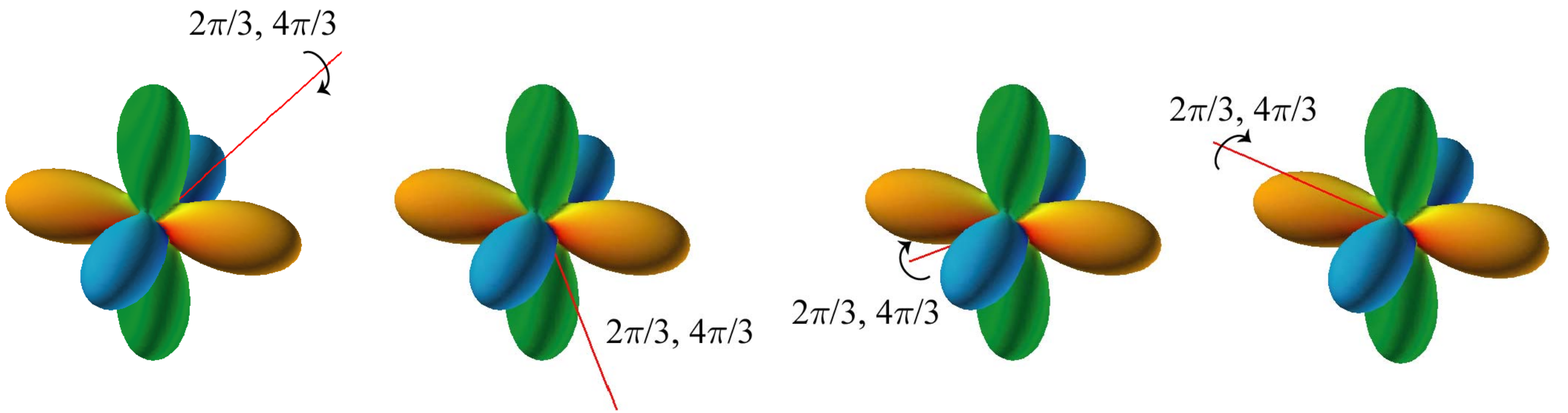
$$\boldsymbol{e}_1 = (1,1,1)/\sqrt{3}, \quad \boldsymbol{e}_2 = (1,-1,-1)/\sqrt{3}$$

$$\boldsymbol{e}_3 = (-1,1,-1)/\sqrt{3}, \quad \boldsymbol{e}_4 = (-1,-1,1)/\sqrt{3}$$

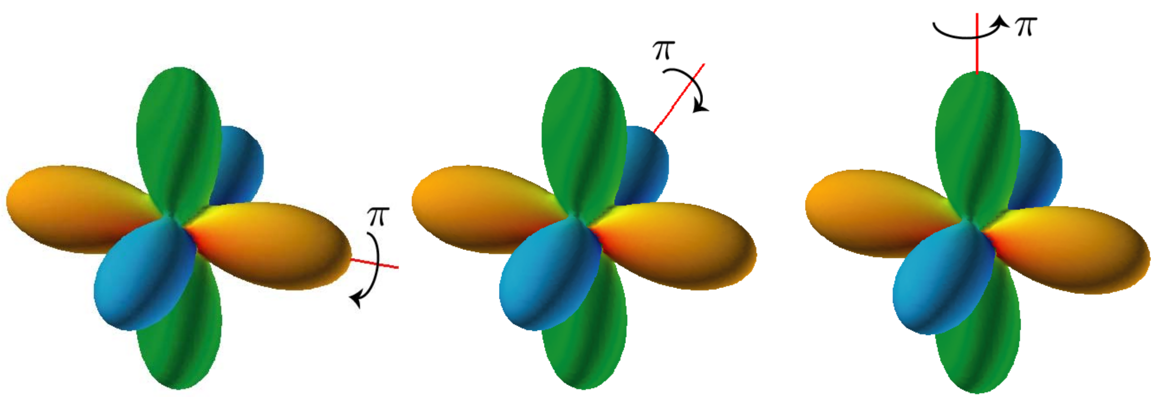
Vortices in Spin-2 BEC

vortices	mass circulation	core structure
gauge	1	density core
Integer spin	0	polar core
1/2 spin	0	polar core
1/3	1/3	ferromagnetic core
2/3	2/3	ferromagnetic core

Topological Charge of Vortices is Non-Abelian



$\boldsymbol{e}_1 = (1,1,1)$ $\boldsymbol{e}_2 = (1,-1,-1)$ $\boldsymbol{e}_3 = (-1,1,-1)$ $\boldsymbol{e}_4 = (-1,-1,1)$

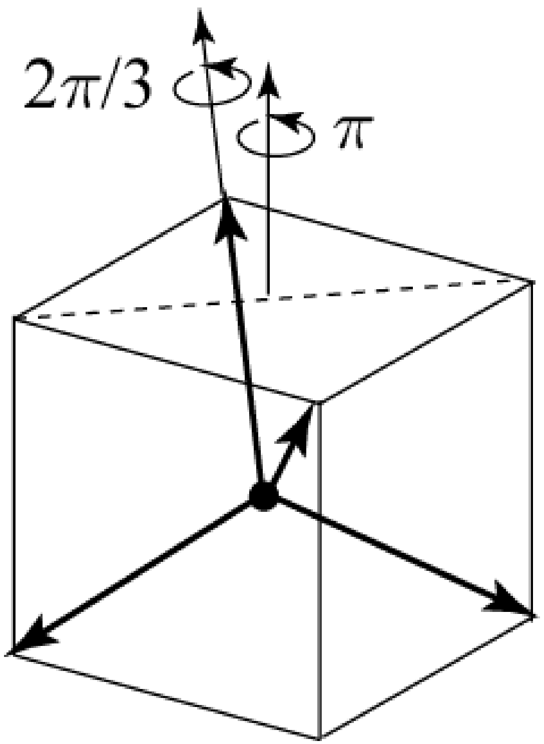


$\boldsymbol{e}_x = (1,0,0)$ $\boldsymbol{e}_y = (0,1,0)$ $\boldsymbol{e}_z = (1,0,0)$

There are 12 rotations for vortices

Non-Abelian Vortices

12 rotations makes non-Abelian tetrahedral group T



Topological charge can be expressed by non-Abelian algebra which includes tetrahedral symmetry
→non-Abelian vortex

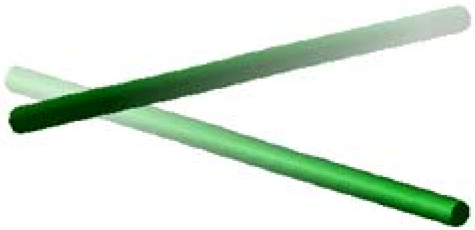
$$\pi_1(G/H) = (Z_2 \ltimes T) \ltimes Z$$

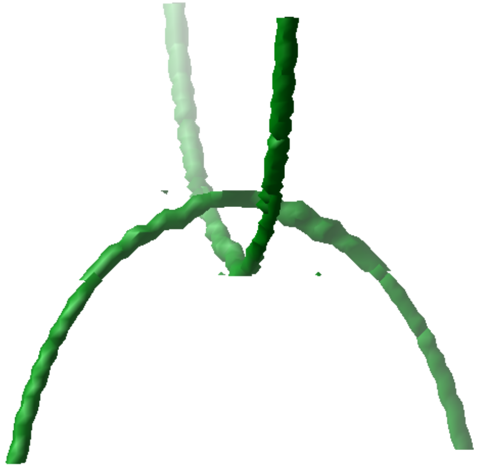
Collision Dynamics of Vortices

"**Non-Abelian**" character becomes remarkable when two vortices collide with each other

→Numerical simulation of the Gross-Pitaevskii equation

Initial state：two straight vortices in oblique angle, linked vortices





Gross-Pitaevskii Equation

$$i\hbar\frac{\partial\Psi_m}{\partial t}=\frac{\delta H}{\delta\Psi_m^*}$$

$$i\hbar\frac{\partial\Psi_2}{\partial t}=-\frac{\hbar^2}{2M}\nabla^2\Psi_2+c_0n_{\mathrm{tot}}\Psi_2+c_1(F_-\Psi_1+2F_z\Psi_2)+\frac{c_2}{\sqrt{5}}A_{00}\Psi^*_{-2}$$

$$i\hbar\frac{\partial\Psi_1}{\partial t}=-\frac{\hbar^2}{2M}\nabla^2\Psi_1+c_0n_{\mathrm{tot}}\Psi_1+c_1\left(\frac{\sqrt{6}}{2}F_-\Psi_0+F_+\Psi_2+F_z\Psi_1\right)-\frac{c_2}{\sqrt{5}}A_{00}\Psi^*_{-1}$$

$$i\hbar\frac{\partial\Psi_0}{\partial t}=-\frac{\hbar^2}{2M}\nabla^2\Psi_0+c_0n_{\mathrm{tot}}\Psi_0+\frac{\sqrt{6}}{2}c_1(F_-\Psi_{-1}+F_+\Psi_1)+\frac{c_2}{\sqrt{5}}A_{00}\Psi^*_0$$

$$i\hbar\frac{\partial\Psi_{-1}}{\partial t}=-\frac{\hbar^2}{2M}\nabla^2\Psi_{-1}+c_0n_{\mathrm{tot}}\Psi_{-1}+c_1\left(\frac{\sqrt{6}}{2}F_+\Psi_0+F_-\Psi_{-2}-F_z\Psi_{-1}\right)-\frac{c_2}{\sqrt{5}}A_{00}\Psi^*_1$$

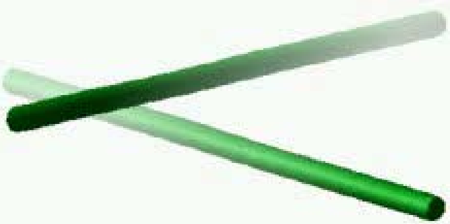
$$i\hbar\frac{\partial\Psi_{-2}}{\partial t}=-\frac{\hbar^2}{2M}\nabla^2\Psi_{-2}+c_0n_{\mathrm{tot}}\Psi_{-2}+c_1(F_+\Psi_{-1}-2F_z\Psi_{-2})+\frac{c_2}{\sqrt{5}}A_{00}\Psi^*_2$$

Used Pair of Vortices

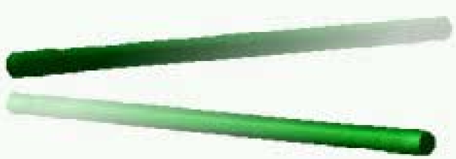
1, same vortices	1/3 vortex (e_1)	1/3 vortex (e_1)
2, different commutative vortices	1/3 vortex (e_1)	2/3 vortex (e_1)
3, different non-commutative vortices	1/3 vortex (e_1)	2/3 vortex (e_2)
	1/3 vortex (e_1)	1/3 vortex (e_2)

Collision Dynamics of Vortices

Commutative topological charge

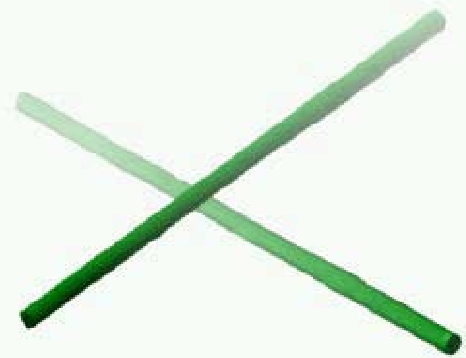


reconnection

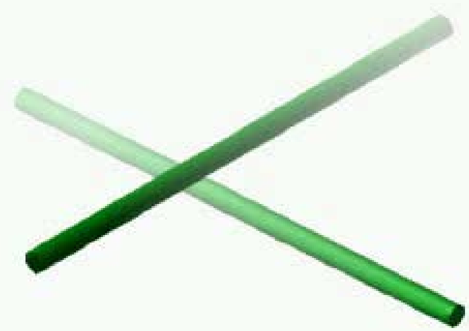


passing through

Non-commutative topological charge



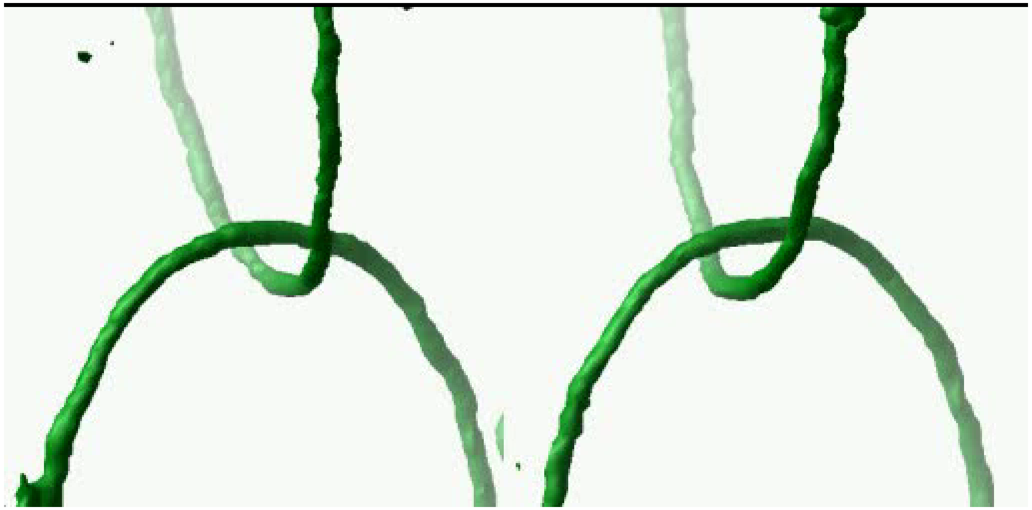
polar rung



ferromagnetic rung

Collision Dynamics of Linked Vortices

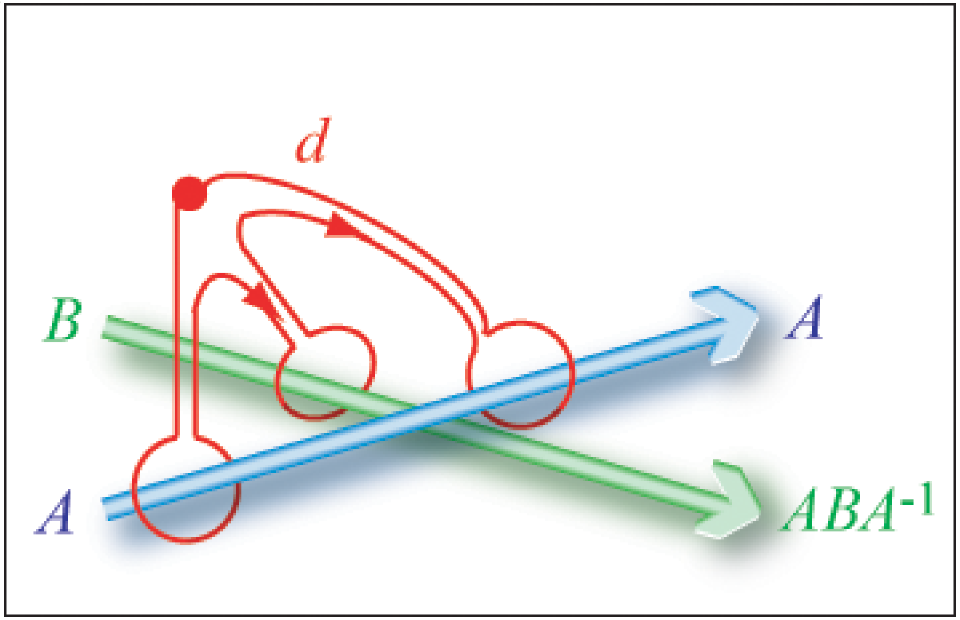
Commutative Non-commutative



untangle not untangle

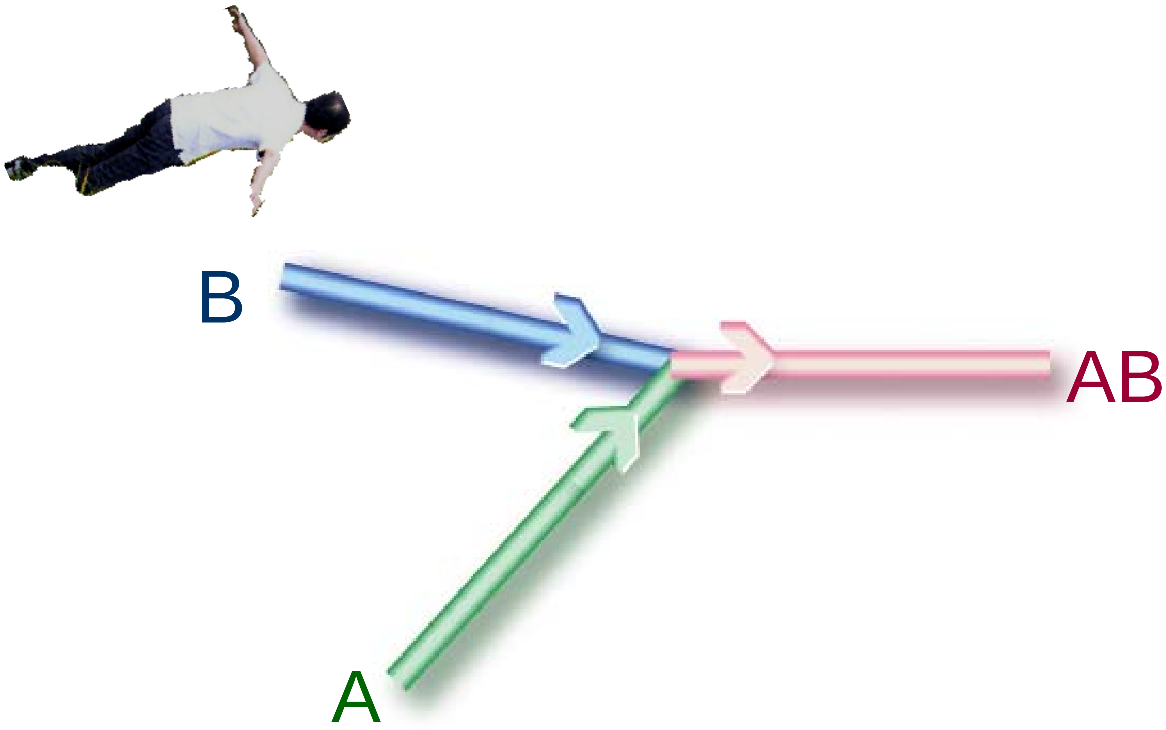
Algebraic Approach

Consider 4 closed paths encircling two vortices

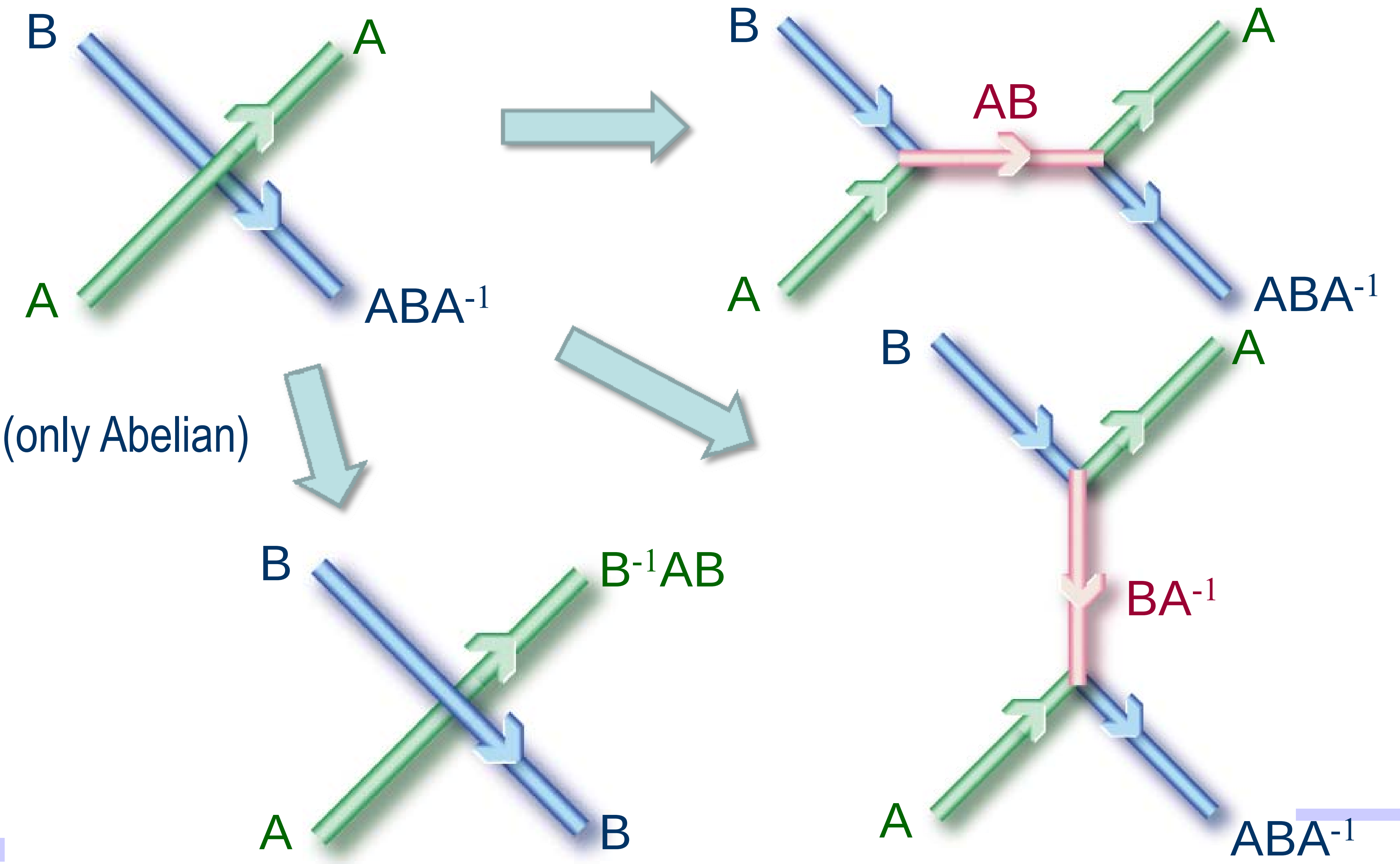


Path d defines vortex B as ABA^{-1} (same conjugacy class)

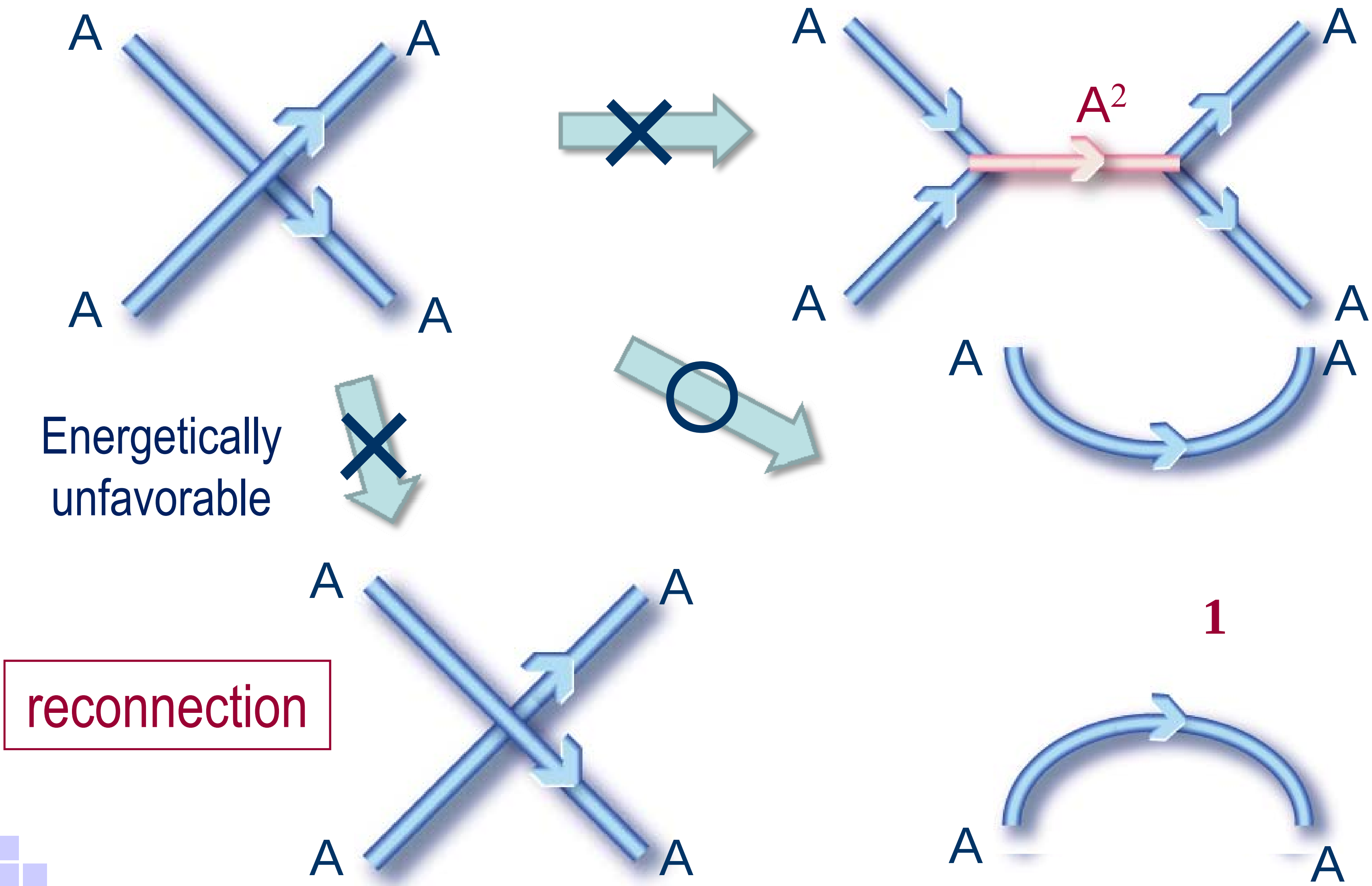
Y-shape Junction



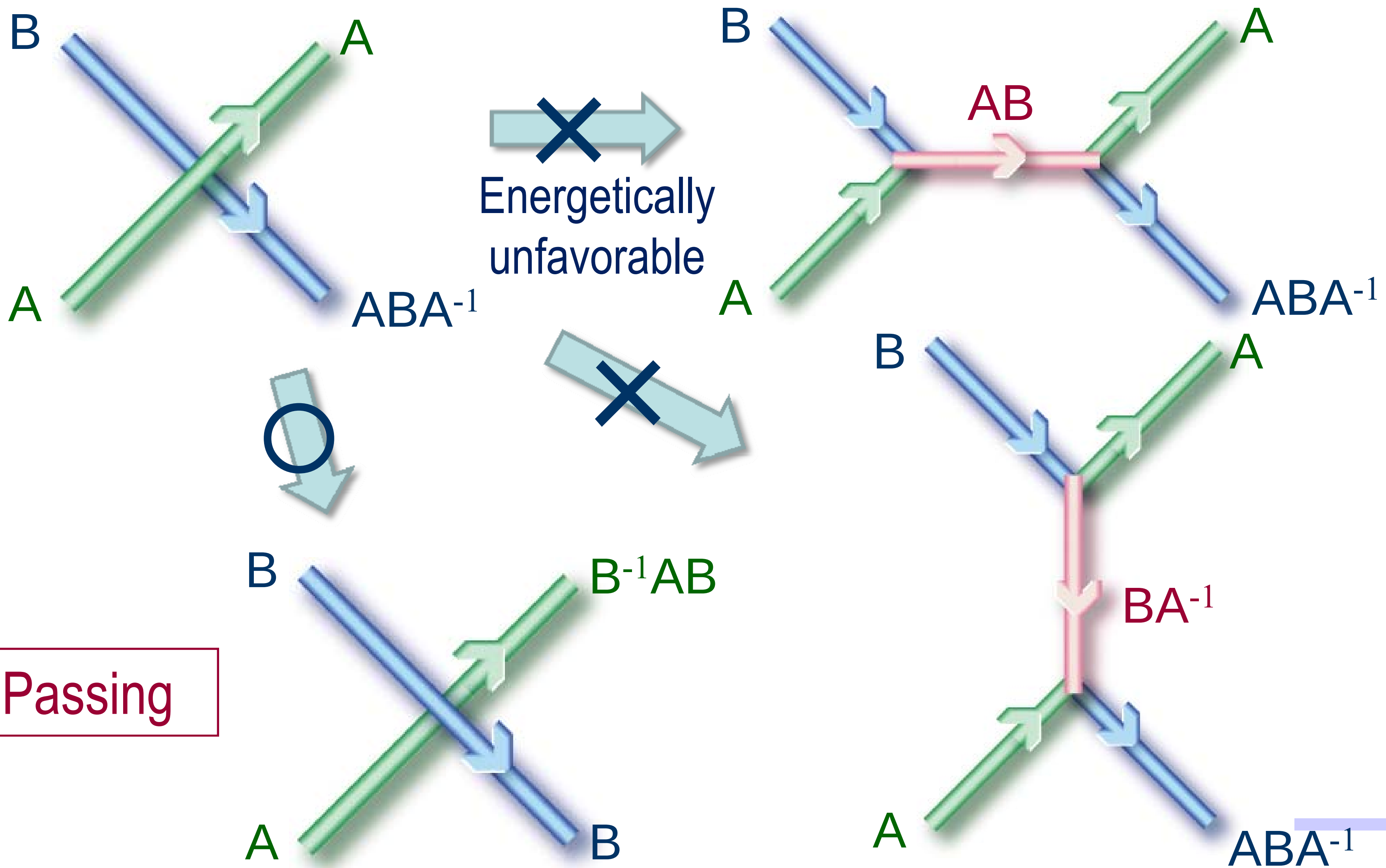
Collision of Vortices



Collision of Same Vortices

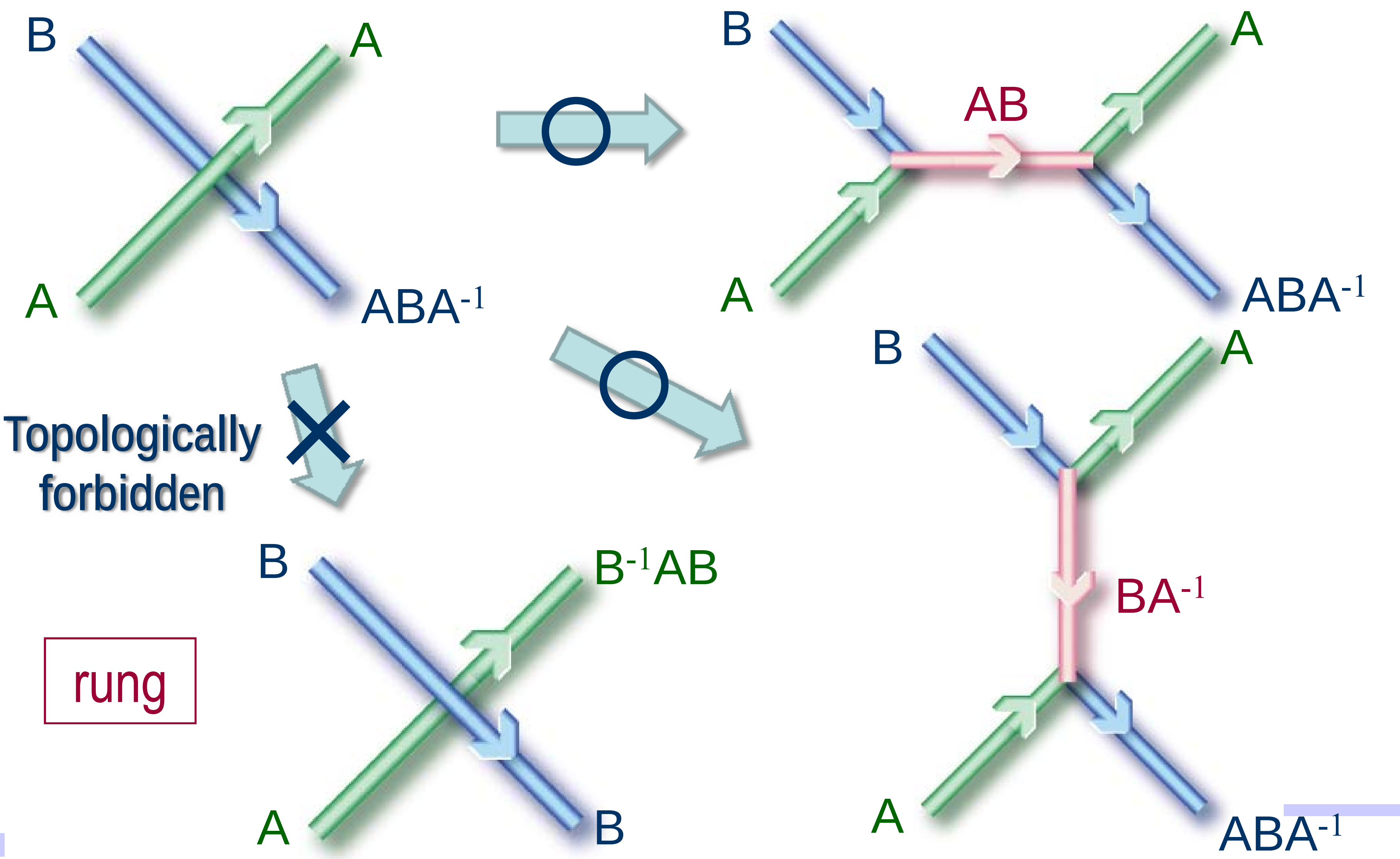


Collision of Different Commutative Vortices

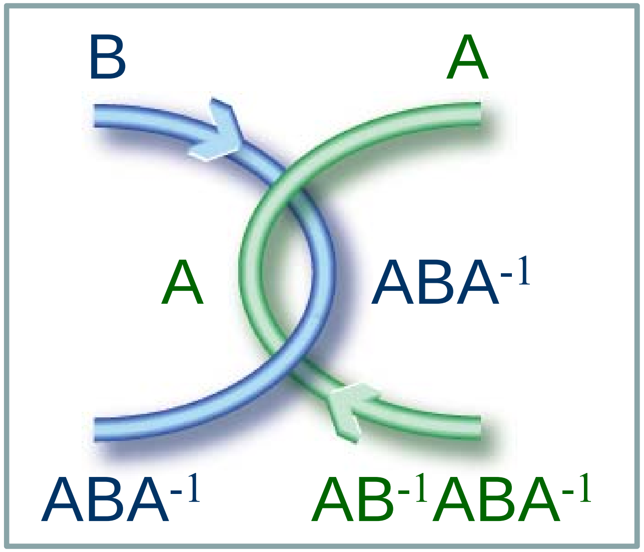


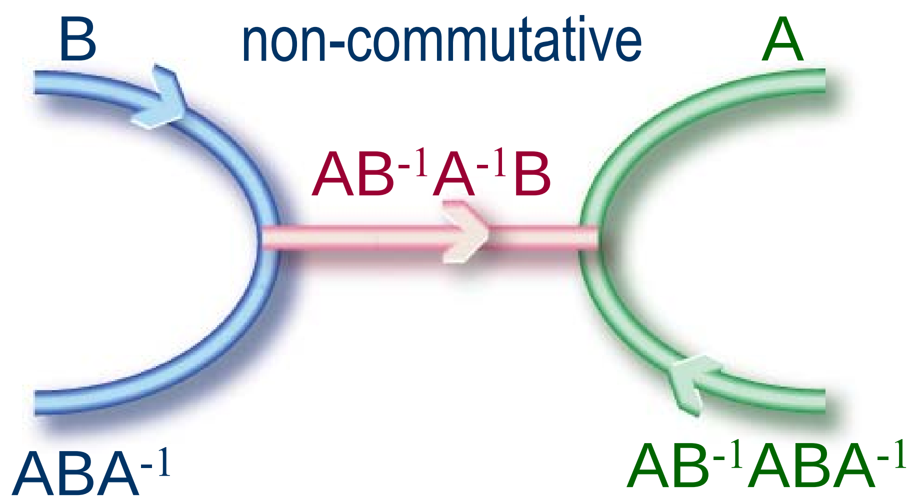
Passing

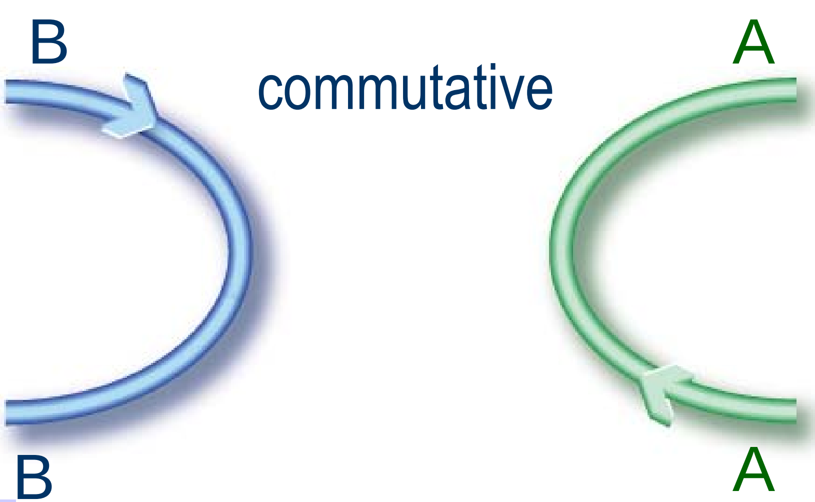
Collision of Different Non-commutative Vortices



Linked Vortices





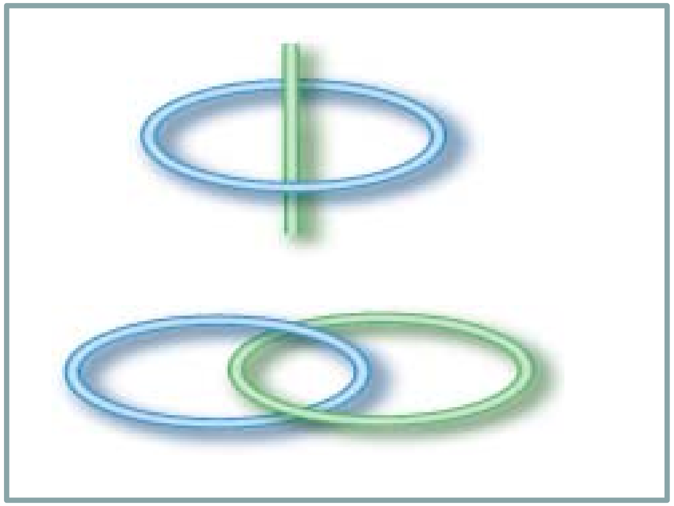


Linked vortices cannot untangle

Summary

1. Vortices with non-commutative circulations are defined as non-Abelian vortices.
2. Non-Abelian vortices can be realized in the cyclic phase of spin-2 BEC
3. Collision of two non-Abelian vortices create a new vortex between them as a rung (networking structure).

Future: Topological Charge of Linked Vortices

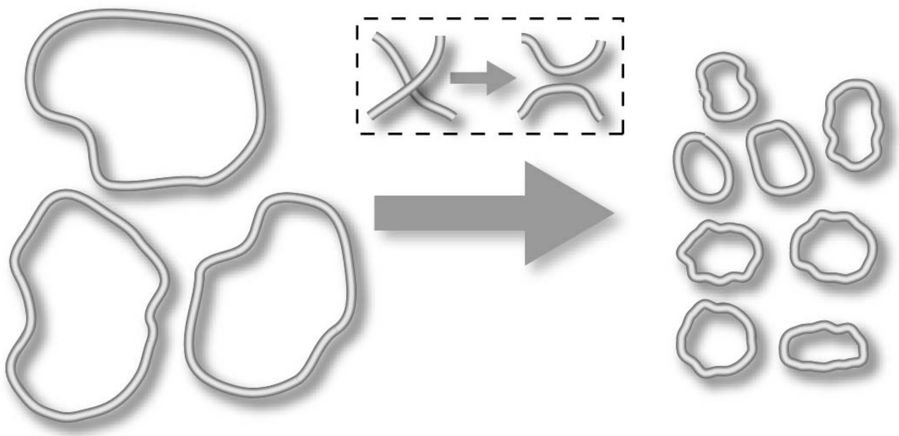


$\neq$

Linked vortex itself has another topological charge
→Searching and applying new homotopy theories

Poster-11, S. Kobayashi "Classification of topological defects by Fox homotopy group"

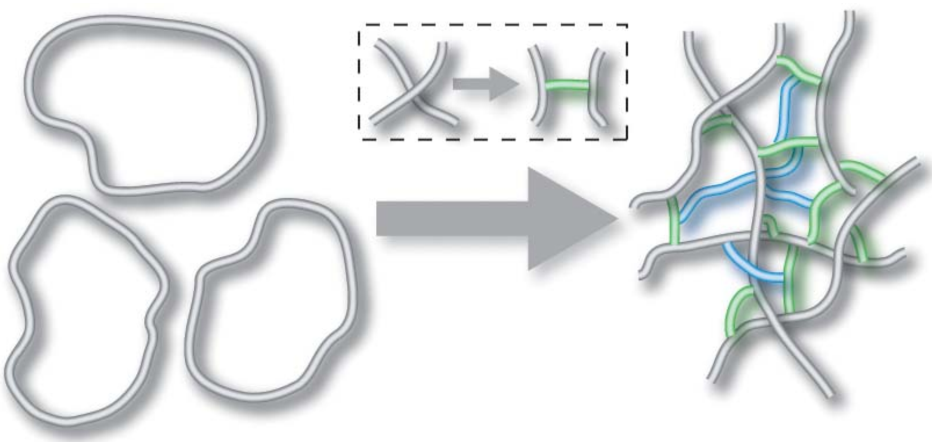
Future: Network Structure in Quantum Turbulence



Turbulence with Abelian vortices

↓

•Cascade of vortices



Turbulence with non-Abelian vortices

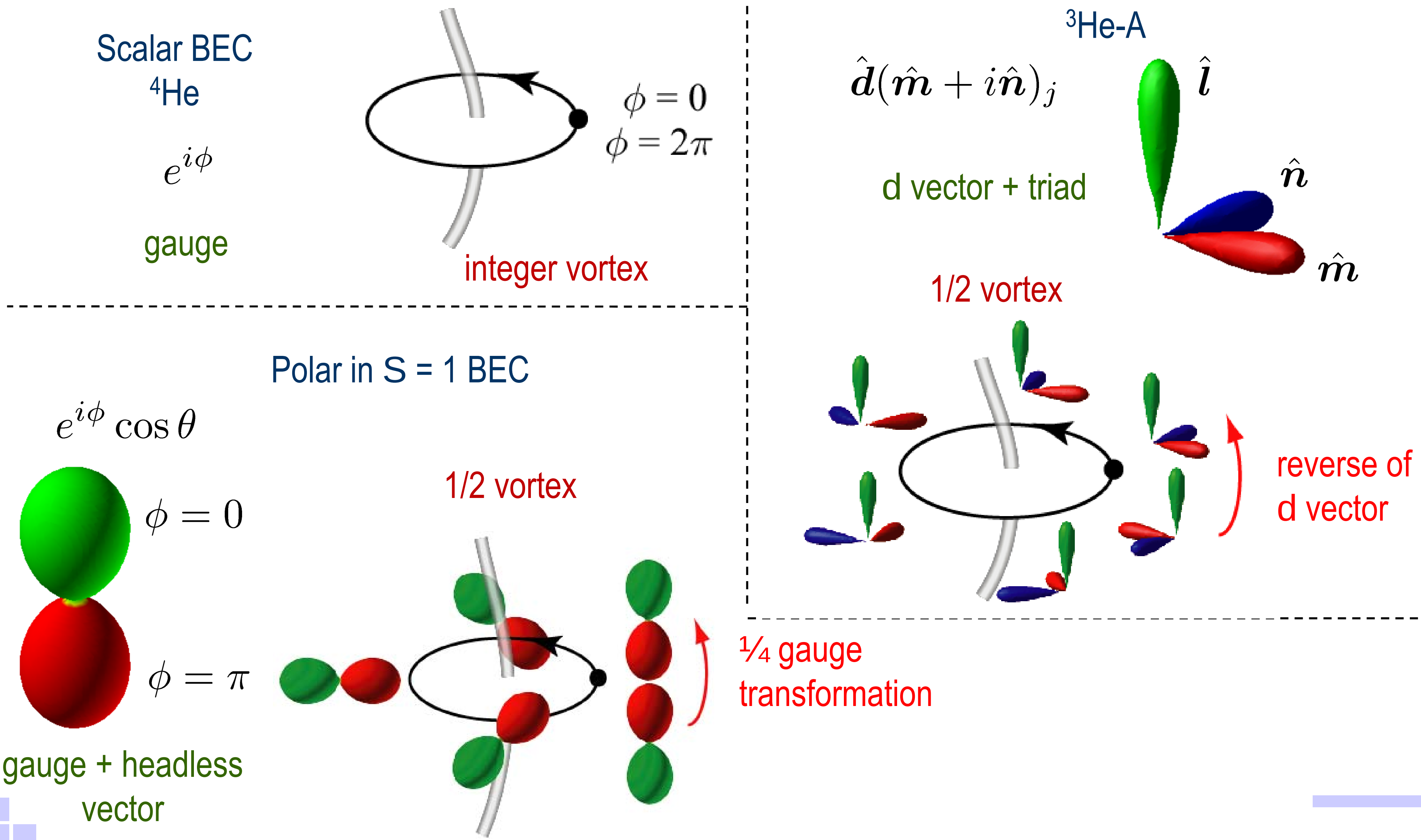
↓

•Large-scale networking structures among vortices with rungs

•Non-cascading turbulence

New turbulence!

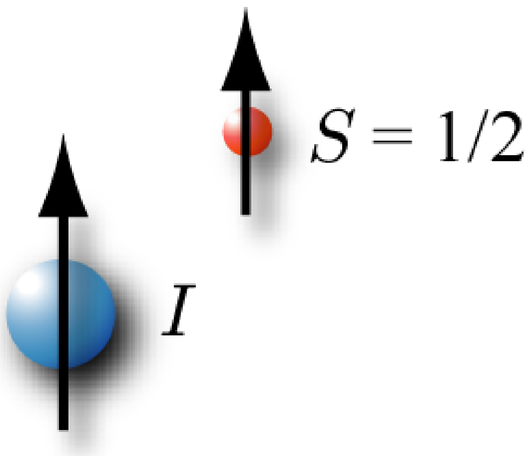
Quantized Vortices in Multi-component BEC



Spin-2 BEC

Bose-Einstein condensate in optical trap
(spin degrees of freedom is alive)

Hyperfine coupling
$(F = I + S)$



^{87}Rb $(I = 3/2)$

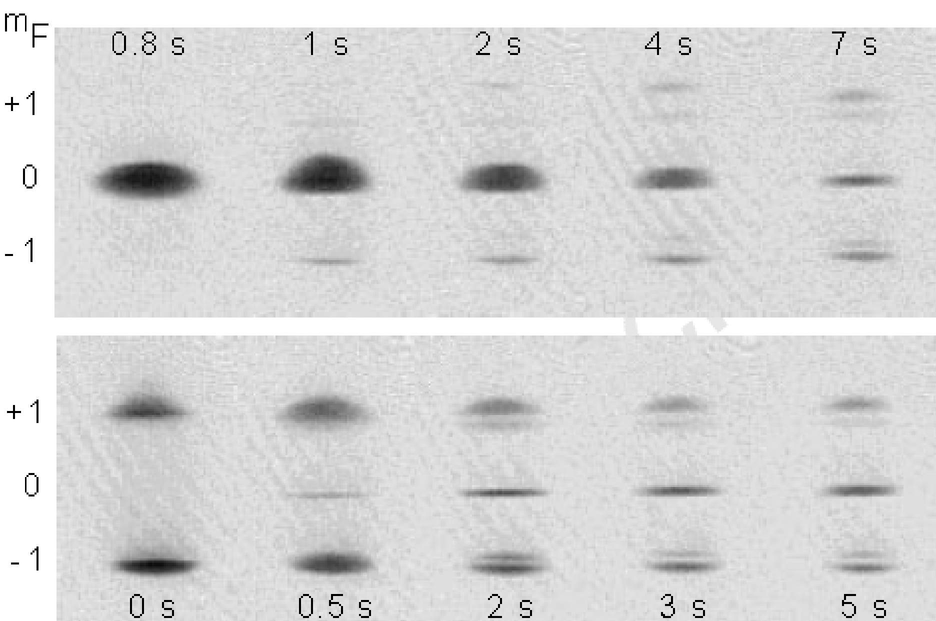
$$F = 2 \begin{cases} m_F = 2 \\ m_F = 1 \\ m_F = 0 \\ m_F = -1 \\ m_F = -2 \end{cases} \qquad F = 1 \begin{cases} m_F = 1 \\ m_F = 0 \\ m_F = -1 \end{cases}$$

BEC characterized by m_F

Spin dynamics of BEC

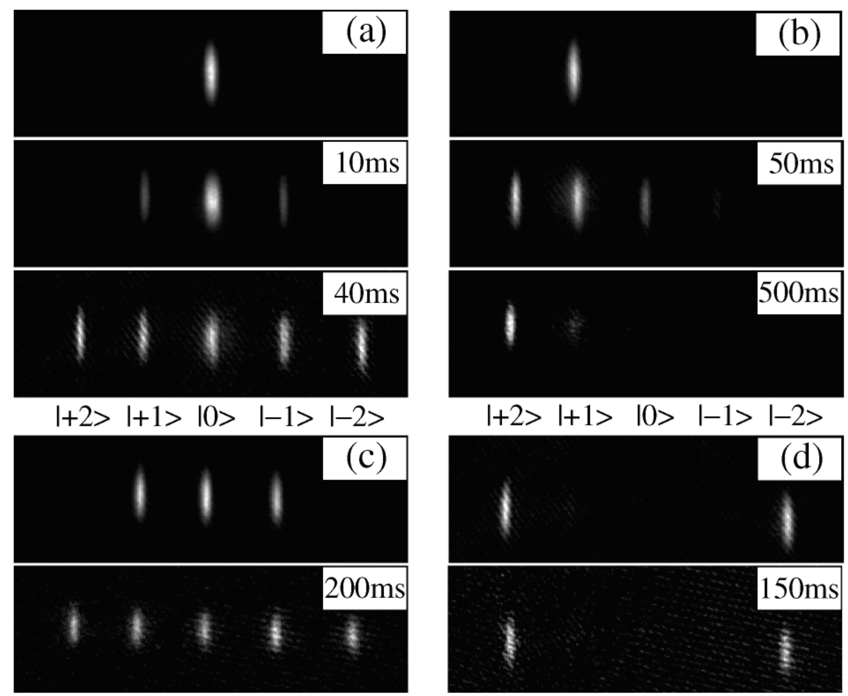
Stern-Gerlach experiment

F = 1



J. Stenger et al. Nature 396, 345 (1998)

F = 2



H. Schmaljohann et al. PRL 92, 040402 (2004)

Spin-2 BEC

$$H \simeq \int d\boldsymbol{x} \left[-\Psi_m^* \frac{\hbar^2}{2M} \nabla^2 \Psi_m + (mp + m^2 q) n_{\rm tot} + \frac{c_0}{2} n_{\rm tot}^2 + \frac{c_1}{2} \boldsymbol{F}^2 + \frac{c_2}{2} |A_{00}|^2 \right]$$

1. $c_1 < 0 \longrightarrow$ ferromagnetic phase : $F \neq 0$
2. $c_1 > 0$, $c_2 < 0 \longrightarrow$ polar phase : $F = 0$, $A_{00} \neq 0$
3. $c_1 > 0$, $c_2 > 0 \longrightarrow$ cyclic phase : $F = A_{00} = 0$

ferromagnetic

$$e^{i\phi} e^{-i\boldsymbol{e}\cdot\hat{\boldsymbol{F}}\alpha} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

polar

$$e^{i\phi} e^{-i\boldsymbol{e}\cdot\hat{\boldsymbol{F}}\alpha} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

cyclic

$$e^{i\phi} e^{-i\boldsymbol{e}\cdot\hat{\boldsymbol{F}}\alpha} \begin{pmatrix} i/2 \\ 0 \\ 1/\sqrt{2} \\ 0 \\ i/2 \end{pmatrix}$$

Spin-2 BEC

$$H \simeq \int d\boldsymbol{x} \left[-\Psi_m^* \frac{\hbar^2}{2M} \nabla^2 \Psi_m + (mp + m^2 q) n_{\mathrm{tot}} + \frac{c_0}{2} n_{\mathrm{tot}}^2 + \frac{c_1}{2} \boldsymbol{F}^2 + \frac{c_2}{2} |A_{00}|^2 \right]$$

1. $c_1 < 0 \longrightarrow$ ferromagnetic phase : $F \neq 0$
2. $c_1 > 0,\ c_2 < 0 \longrightarrow$ polar phase : $F = 0,\ A_{00} \neq 0$
3. $c_1 > 0,\ c_2 > 0 \longrightarrow$ cyclic phase : $F = A_{00} = 0$

Experimental observation for ^{87}Rb

$c_1 / (4\pi\hbar^2 / M) = (0.99 \pm 0.06)\ a_B$

$c_2 / (4\pi\hbar^2 / M) = (-0.53 \pm 0.58)\ a_B$

Whether the system is in polar or cyclic has not decided yet

A. Widera et al. New J. Phys 8, 152 (2006)

Phase Diagram

$$H \simeq \int dx \left[-\Psi_m^* \frac{\hbar^2}{2M} \nabla^2 \Psi_m + (mp + m^2 q) n_{\rm tot} + \frac{c_0}{2} n_{\rm tot}^2 + \frac{c_1}{2} \boldsymbol{F}^2 + \frac{c_2}{2} |A_{00}|^2 \right]$$

$$\frac{E_{\rm f}}{N} = 2p + 4q + \frac{c_0 n_{\rm tot}}{2} + 2 c_1 n_{\rm tot}$$

$$\frac{E_{\rm pu}}{N} = \frac{c_0 n_{\rm tot}}{2} + \frac{c_2 n_{\rm tot}}{10}$$

$$\frac{E_{\rm pb}}{N} = 4q + \frac{c_0 n_{\rm tot}}{2} + \frac{c_2 n_{\rm tot}}{10}$$

$$\frac{E_{\rm c}}{N} = 2q + \frac{c_0 n_{\rm tot}}{2}$$

Phase diagram with neglecting linear Zeeman

$$B_{\rm f-pu} : q = -\frac{c_1 n_{\rm tot}}{2} + \frac{c_2 n_{\rm tot}}{40}$$

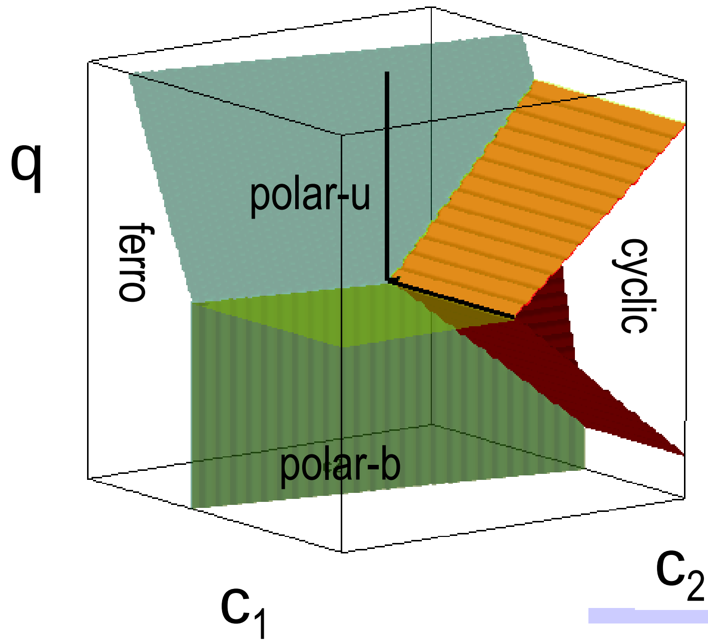
$$B_{\rm f-pb} : c_1 n_{\rm tot} = \frac{c_2 n_{\rm tot}}{20}$$

$$B_{\rm f-c} : q = -c_1 n_{\rm tot}$$

$$B_{\rm pu-pb} : q = 0$$

$$B_{\rm pu-c} : q = \frac{c_2 n_{\rm tot}}{20}$$

$$B_{\rm pb-c} : q = -\frac{c_2 n_{\rm tot}}{20}$$



Phase Diagram

$$H \simeq \int dx \left[-\Psi_m^* \frac{\hbar^2}{2M} \nabla^2 \Psi_m + (mp + m^2 q) n_{\text{tot}} + \frac{c_0}{2} n_{\text{tot}}^2 + \frac{c_1}{2} \boldsymbol{F}^2 + \frac{c_2}{2} |A_{00}|^2 \right]$$